Knuth-Bendix Completion: Outlook

Classical completion:

Fails, if an equation can neither be oriented nor deleted.

Unfailing Completion:

Use an ordering $>$ that is total on ground terms.

If an equation cannot be oriented, use it in both directions for rewriting (except if that would yield a larger term).
In other words, consider the relation $\leftrightarrow_E \cap \not\leq$.

Special case of superposition (see next chapter).

7 Superposition

Recapitulation: Equational Clauses

Atom: either $p(s_1, \ldots, s_m)$ with $p/m \in \Pi$ or $s \approx t$.

Literal: Atom or negated atom.

Clause: (possibly empty) disjunction of literals
(all variables implicitly universally quantified).

For refutational theorem proving, it is sufficient to consider sets of clauses:
every first-order formula F can be translated into a set of clauses N such that F is unsatisfiable if and only if N is unsatisfiable.

The Resolution Calculus

In the non-equational case, unsatisfiability can for instance be checked using the (ordered) resolution calculus.

Notation:

An inference rule

$$\frac{C_n, \ldots, C_1}{C_0}$$

means:

add C_0 to the current set of formulas N, if N contains $C_n, \ldots, C_1$ ($C_n, \ldots, C_1$ are not deleted!).

The Resolution Calculus

(Ordered) resolution: inference rules:

	Ground case:	Non-ground case:
Resolution:	$$\frac{D' \vee A \qquad C' \vee \neg A}{D' \vee C'}$$	$$\frac{D' \vee A \qquad C' \vee \neg A'}{(D' \vee C')\sigma}$$ where $\sigma = \mathrm{mgu}(A, A')$.
Factoring:	$$\frac{C' \vee A \vee A}{C' \vee A}$$	$$\frac{C' \vee A \vee A'}{(C' \vee A)\sigma}$$ where $\sigma = \mathrm{mgu}(A, A')$.

The Resolution Calculus

Ordering restrictions:

Let $>$ be a well-founded ordering on atoms that is stable under substitutions and total on ground atoms.

Literal ordering $>_L$:
compares literals by comparing lexicographically first the respective atoms using $>$ and then their polarities (negative $>$ positive).

Clause ordering $>_C$:
compares clauses by comparing their multisets of literals using the multiset extension of $>_L$.

The Resolution Calculus

Ordering restrictions:

Perform an inference only if the literals that are involved in the inferences ($[\neg]\, A$, $[\neg]\, A'$) are maximal in the respective clauses
(strictly maximal for the left premise of Resolution).

Further restrictions are possible (later).

The Resolution Calculus

Resolution is (even with ordering restrictions) refutationally complete:

Dynamic view of refutational completeness:

If N is unsatisfiable ($N \models \bot$) then fair* derivations from N produce $\bot$.

Static view of refutational completeness:

If N is saturated*, then N is unsatisfiable if and only if $\bot \in N$.

* to be made precise later.

Resolution: Refutational Completeness

Proving refutational completeness for the ground case:

We have to show:

If N is saturated (i. e., if sufficiently many inferences have been computed), and $\bot \notin N$, then N is satisfiable (i. e., has a model).

Resolution: Refutational Completeness

A Σ-interpretation (Σ-algebra) $\mathcal{B}$ is called a Herbrand interpretation, if

$U_{\mathcal{B}} = \mathsf{T}_{\Sigma}(\emptyset)$,

$f_{\mathcal{B}}(t_1, \ldots, t_n) = f(t_1, \ldots, t_n)$.

Properties of Herbrand interpretations:

Ground terms are interpreted by themselves;
n-ary predicates are interpreted by subsets of $(\mathsf{T}_{\Sigma}(\emptyset))^n$.

Every Herbrand interpretation $\mathcal{B}$ corresponds to a set of ground atoms I: if $A \in I$ then A is true and $\neg A$ is false in $\mathcal{B}$; if $A \notin I$ then A is false and $\neg A$ is true in $\mathcal{B}$.

Resolution: Refutational Completeness

Model construction:

Suppose that N be saturated and $\bot \notin N$.
We inspect all clauses in N in ascending order and construct a sequence of Herbrand interpretations
(starting with the empty interpretation: all atoms are false).

If a clause C is false in the current interpretation, and has a positive and strictly maximal literal A, then extend the current interpretation such that C becomes true: add A to the current interpretation. (Then C is called productive.)

Otherwise, leave the current interpretation unchanged.

Resolution: Refutational Completeness

The sequence of interpretations has the following properties:

(1) If an atom is true in some interpretation, then it remains true in all future interpretations.

(2) If a clause is true at the time where it is inspected, then it remains true in all future interpretations.

(3) If a clause $C = C' \vee A$ is productive, then C remains true and C' remains false in all future interpretations.

Show by induction: if N is saturated and $\bot \notin N$, then every clause in N is either true at the time where it is inspected or productive.

Resolution: Refutational Completeness

Note:
For the induction proof, it is not necessary that the conclusion of an inference is contained in N.
If is sufficient that it follows from clauses that are smaller than the largest (i. e., rightmost) premise.

A ground inference is called redundant w. r. t. N, if its conclusion follows from clauses in N that are smaller than the largest (i. e., rightmost) premise.

N is called saturated, if every inference from clauses in N is redundant w. r. t. N.

Resolution: Refutational Completeness

Proving refutational completeness for the non-ground case:

If $C_i\theta$ is a ground instance of the clause C_i for $i \in \{0, \ldots, n\}$ and

$$\frac{C_n, \ldots, C_1}{C_0}$$

and

$$\frac{C_n\theta, \ldots, C_1\theta}{C_0\theta}$$

are inferences, then the latter inference is called a ground instance of the former.

Resolution: Refutational Completeness

For a set N of clauses, let $\overline{N}$ be the set of all ground instances of clauses in N.

An inference is redundant w. r. t. N, if all its ground instances are redundant w. r. t. $\overline{N}$.

Construct the interpretation from the set $\overline{N}$ of all ground instances of clauses in N:

N is saturated and does not contain $\bot$

$\Rightarrow$ $\overline{N}$ is saturated and does not contain $\bot$

$\Rightarrow$ $\overline{N}$ has a Herbrand model I

$\Rightarrow$ I is a model of N.

The Superposition Calculus

Conventions:

From now on: $\Pi = \emptyset$ (equality is the only predicate).

Inference rules are to be read modulo symmetry of the equality symbol.

The Superposition Calculus

Ground inference rules:

Pos. Superposition:
$$\frac{D' \lor t \approx t' \qquad C' \lor s[t] \approx s'}{D' \lor C' \lor s[t'] \approx s'}$$

Neg. Superposition:
$$\frac{D' \lor t \approx t' \qquad C' \lor s[t] \not\approx s'}{D' \lor C' \lor s[t'] \not\approx s'}$$

Equality Resolution:
$$\frac{C' \lor s \not\approx s}{C'}$$

(Note: We will need one further inference rule.)

The Superposition Calculus

Ordering restrictions:

Some considerations:

The literal ordering must depend primarily on the larger term of an equation.

As in the resolution case, negative literals must be a bit larger than the corresponding positive literals.

Additionally, we need the following property:
If $s > t > u$, then $s \not\approx u$ must be larger than $s \approx t$.
In other words, we must compare first the larger term, then the polarity, and finally the smaller term.

The Superposition Calculus

The following construction has the required properties:

Let $>$ be a reduction ordering that is total on ground terms.

To a positive literal $s \approx t$, we assign the multiset $\{s, t\}$, to a negative literal $s \not\approx t$ the multiset $\{s, s, t, t\}$. The literal ordering $>_L$ compares these multisets using the multiset extension of $>$.

The clause ordering $>_C$ compares clauses by comparing their multisets of literals using the multiset extension of $>_L$.

The Superposition Calculus

Ordering restrictions:

Perform an inference only if

- the literals that are involved in the inferences are maximal in the respective clauses (strictly maximal for positive literals in superposition inferences), and
- in these literals, the lhs is greater than or equal to the rhs (strictly greater in superposition inferences).

The Superposition Calculus

Model construction:

We want to use roughly the same ideas as in the completeness proof for resolution.

But: a Herbrand interpretation does not work for equality:
The equality symbol $\approx$ must be interpreted by equality in the interpretation.

The Superposition Calculus

Solution: Define a congruence relation on $\mathrm{T}_\Sigma(\emptyset)$ and take $\mathrm{T}_\Sigma(\emptyset)$ modulo this congruence relation as the universe.

Then two terms are equal in the interpretation, if and only if they are in the congruence relation.

If the congruence relation is defined by a terminating and confluent rewrite system R, then two terms s and t are equal in the interpretation, if and only if $s \downarrow_R t$.