5 Termination

Termination

Termination problems:

Given a finite TRS R and a term t, are all R-reductions starting from t terminating?

Given a finite TRS R, are all R-reductions terminating?

Proposition:
Both termination problems for TRSs are undecidable in general.

Proof:
Encode Turing machines using rewrite rules and reduce the (uniform) halting problems for TMs to the termination problems for TRSs.

Termination

Consequence:

Decidable criteria for termination are not complete.

Reduction Orderings

Goal:

Given a finite TRS R, show termination of R by looking at finitely many rules $l \to r \in R$, rather than at infinitely many possible replacement steps $s \to_R s'$.

Reduction Orderings

A binary relation $\sqsupset$ over $\mathrm{T}_\Sigma(X)$ is called
compatible with Σ-operations,
if $s \sqsupset s'$ implies $f(t_1, \dots, s, \dots, t_n) \sqsupset f(t_1, \dots, s', \dots, t_n)$
for all $f/n \in \Omega$ and $s, s', t_i \in \mathrm{T}_\Sigma(X)$.

Lemma:
The relation $\sqsupset$ is compatible with Σ-operations, if and only if
$s \sqsupset s'$ implies $t[s]_p \sqsupset t[s']_p$
for all $s, s', t \in \mathrm{T}_\Sigma(X)$ and $p \in \mathrm{Pos}(t)$.

(compatible with Σ-operations = compatible with Σ-contexts)

Reduction Orderings

A binary relation $\sqsupset$ over $\mathrm{T}_\Sigma(X)$ is called stable under substitutions, if $s \sqsupset s'$ implies $s\sigma \sqsupset s'\sigma$ for all $s, s' \in \mathrm{T}_\Sigma(X)$ and substitutions σ.

Reduction Orderings

A binary relation $\sqsupset$ is called a rewrite relation, if it is compatible with Σ-operations and stable under substitutions.

Example: If R is a TRS, then $\rightarrow_R$ is a rewrite relation.

A strict partial ordering over $\mathrm{T}_\Sigma(X)$ that is a rewrite relation is called rewrite ordering.

A well-founded rewrite ordering is called reduction ordering.

Reduction Orderings

Theorem:

A TRS R terminates if and only if there exists a reduction ordering $>$ such that $l > r$ for every rule $l \rightarrow r \in R$.

Proof:

"if": $s \rightarrow_R s'$ if and only if $s = t[l\sigma]_p$, $s' = t[r\sigma]_p$. Now $l > r$ implies $l\sigma > r\sigma$ and therefore $t[l\sigma]_p > t[r\sigma]_p$. So $\rightarrow_R \subseteq >$. Since $>$ is a well-founded ordering, $\rightarrow_R$ is terminating.

"only if": Define $> = \rightarrow_R^+$. If $\rightarrow_R$ is terminating, then $>$ is a reduction ordering.

The Interpretation Method

Proving termination by interpretation:

Let $\mathcal{A}$ be a Σ-algebra;

let $>$ be a well-founded strict partial ordering on its universe.

Define the ordering $>_{\mathcal{A}}$ over $\mathrm{T}_{\Sigma}(X)$ by $s >_{\mathcal{A}} t$ iff $\mathcal{A}(\alpha)(s) > \mathcal{A}(\alpha)(t)$ for all assignments $\alpha : X \to U_{\mathcal{A}}$.

Is $>_{\mathcal{A}}$ a reduction ordering?

The Interpretation Method

Lemma:

$>_{\mathcal{A}}$ is stable under substitutions.

Proof:

Let $s >_{\mathcal{A}} s'$, that is,
$\mathcal{A}(\alpha)(s) > \mathcal{A}(\alpha)(s')$ for all assignments $\alpha : X \to U_{\mathcal{A}}$.
Let σ be a substitution. We have to show that
$\mathcal{A}(\beta)(s\sigma) > \mathcal{A}(\beta)(s'\sigma)$ for all assignments $\beta : X \to U_{\mathcal{A}}$.
Define $\alpha(x) = \mathcal{A}(\beta)(x\sigma)$,
then $\mathcal{A}(\alpha)(t) = \mathcal{A}(\beta)(t\sigma)$ for every $t \in \mathrm{T}_{\Sigma}(X)$.
Thus $\mathcal{A}(\beta)(s\sigma) = \mathcal{A}(\alpha)(s) > \mathcal{A}(\alpha)(s') = \mathcal{A}(\beta)(s'\sigma)$.
Therefore $s\sigma >_{\mathcal{A}} s'\sigma$.

The Interpretation Method

A function $F : U_{\mathcal{A}}^n \to U_{\mathcal{A}}$ is called monotone (w.r.t. $>$),
if $a > a'$ implies
$F(b_1, \ldots, a, \ldots, b_n) > F(b_1, \ldots, a', \ldots, b_n)$
for all $a, a', b_i \in U_{\mathcal{A}}$.

The Interpretation Method

Lemma:
If the interpretation $f_{\mathcal{A}}$ of every function symbol f is monotone w.r.t. $>$, then $>_{\mathcal{A}}$ is compatible with Σ-operations.

Proof:
Let $s > s'$, that is, $\mathcal{A}(\alpha)(s) > \mathcal{A}(\alpha)(s')$ for all $\alpha : X \to U_{\mathcal{A}}$.
Let $\alpha : X \to U_{\mathcal{A}}$ be an arbitrary assignment.
Then $\mathcal{A}(\alpha)(f(t_1, \dots, s, \dots, t_n))$
$= f_{\mathcal{A}}(\mathcal{A}(\alpha)(t_1), \dots, \mathcal{A}(\alpha)(s), \dots, \mathcal{A}(\alpha)(t_n))$
$> f_{\mathcal{A}}(\mathcal{A}(\alpha)(t_1), \dots, \mathcal{A}(\alpha)(s'), \dots, \mathcal{A}(\alpha)(t_n))$
$= \mathcal{A}(\alpha)(f(t_1, \dots, s', \dots, t_n))$.
Therefore $f(t_1, \dots, s, \dots, t_n) >_{\mathcal{A}} f(t_1, \dots, s', \dots, t_n)$.

The Interpretation Method

Theorem:

If the interpretation $f_{\mathcal{A}}$ of every function symbol f is monotone w.r.t. $>$, then $>_{\mathcal{A}}$ is a reduction ordering.

Proof:

By the previous two lemmas, $>_{\mathcal{A}}$ is a rewrite relation.

If there were an infinite chain $s_1 >_{\mathcal{A}} s_2 >_{\mathcal{A}} \ldots$, then it would correspond to an infinite chain $\mathcal{A}(\alpha)(s_1) > \mathcal{A}(\alpha)(s_2) > \ldots$ (with α chosen arbitrarily).

Thus $>_{\mathcal{A}}$ is well-founded.

Irreflexivity and transitivity are proved similarly.

Polynomial Orderings

Polynomial orderings:

Instance of the interpretation method:

The carrier set $U_{\mathcal{A}}$ is some subset of the natural numbers.

To every n-ary function symbol f associate a polynomial $P_f(X_1, \ldots, X_n) \in \mathbb{N}[X_1, \ldots, X_n]$ with coefficients in $\mathbb{N}$ and indeterminates $X_1, \ldots, X_n$.
Then define $f_{\mathcal{A}}(a_1, \ldots, a_n) = P_f(a_1, \ldots, a_n)$ for $a_i \in U_{\mathcal{A}}$.

Polynomial Orderings

Requirement 1:

If $a_1, \ldots, a_n \in U_{\mathcal{A}}$, then $f_A(a_1, \ldots, a_n) \in U_{\mathcal{A}}$.
(Otherwise, $\mathcal{A}$ would not be a Σ-algebra.)

Polynomial Orderings

The mapping from function symbols to polynomials can be extended to terms:
A term t containing the variables $x_1, \ldots, x_n$
yields a polynomial P_t with indeterminates $X_1, \ldots, X_n$
(where X_i corresponds to $\alpha(x_i)$).

Example:

$\Omega = \{a/0,\ f/1,\ g/3\}$,
$U_{\mathcal{A}} = \{\, n \in \mathbb{N} \mid n \geq 1 \,\}$,
$P_a = 3, \quad P_f(X_1) = X_1^2, \quad P_g(X_1, X_2, X_3) = X_1 + X_2 X_3.$

Let $t = g(f(a), f(x), y)$, then $P_t(X, Y) = 9 + X^2 Y$.

Polynomial Orderings

Requirement 2:

f_A must be monotone (w.r.t. $>$).

From now on:

$U_{\mathcal{A}} = \{\, n \in \mathbb{N} \mid n \geq 2 \,\}$.

If $f/0 \in \Omega$, then P_f is a constant ≥ 2.

If $f/n \in \Omega$ with $n \geq 1$, then P_f is a polynomial $P(X_1, \ldots, X_n)$, such that every X_i occurs in some monomial with exponent at least 1 and non-zero coefficient.

$\Rightarrow$ Requirements 1 and 2 are satisfied.

Polynomial Orderings

If P, Q are polynomials in $\mathbb{N}[X_1, \ldots, X_n]$, we write $P > Q$ if $P(a_1, \ldots, a_n) > Q(a_1, \ldots, a_n)$ for all $a_1, \ldots, a_n \in U_{\mathcal{A}}$.

Clearly, $l >_{\mathcal{A}} r$ iff $P_l > P_r$.

Question: Can we check $P_l > P_r$ automatically?

Polynomial Orderings

Hilbert's 10th Problem:

Given a polynomial $P \in \mathbb{Z}[X_1, \ldots, X_n]$ with integer coefficients, is $P = 0$ for some n-tuple of natural numbers?

Theorem:
Hilbert's 10th Problem is undecidable.

Proposition:
Given a polynomial interpretation and two terms l, r, it is undecidable whether $P_l > P_r$.

Proof:
By reduction of Hilbert's 10th Problem.

Polynomial Orderings

One possible solution:

Test whether $P_l(a_1, \ldots, a_n) > P_r(a_1, \ldots, a_n)$ for all $a_1, \ldots, a_n \in \{ x \in \mathbb{R} \mid x \geq 2 \}$.

This is decidable (but very slow).
Since $U_{\mathcal{A}} \subseteq \{ x \in \mathbb{R} \mid x \geq 2 \}$, it implies $P_l > P_r$.

Polynomial Orderings

Another solution (Ben Cherifa and Lescanne):

Consider the difference $P_l(X_1, \ldots, X_n) - P_r(X_1, \ldots, X_n)$ as a polynomial with real coefficients and apply the following inference system to it to show that it is positive for all $a_1, \ldots, a_n \in U_{\mathcal{A}}$:

Polynomial Orderings

$P \ \Rightarrow_{BCL} \ \top,$

if P contains at least one monomial with a positive coefficient and no monomial with a negative coefficient.

$P + c\,X_1^{p_1} \cdots X_n^{p_n} - d\,X_1^{q_1} \cdots X_n^{q_n} \ \Rightarrow_{BCL} \ P + c'\,X_1^{p_1} \ldots X_n^{p_n},$

if $c, d > 0$, $p_i \geq q_i$ for all i,
and $c' = c - d \cdot 2^{(q_1 - p_1) + \cdots + (q_n - p_n)} \geq 0$.

$P + c\,X_1^{p_1} \cdots X_n^{p_n} - d\,X_1^{q_1} \cdots X_n^{q_n} \ \Rightarrow_{BCL} \ P - d'\,X_1^{q_1} \ldots X_n^{q_n},$

if $c, d > 0$, $p_i \geq q_i$ for all i,
and $d' = d - c \cdot 2^{(p_1 - q_1) + \cdots + (p_n - q_n)} > 0$.

Polynomial Orderings

Lemma:

If $P \Rightarrow_{BCL} P'$, then $P(a_1, \ldots, a_n) \geq P'(a_1, \ldots, a_n)$ for all $a_1, \ldots, a_n \in U_{\mathcal{A}}$.

Proof:

Follows from the fact that $a_i \in U_{\mathcal{A}}$ implies $a_i \geq 2$.

Proposition:

If $P \Rightarrow^+_{BCL} \top$, then $P(a_1, \ldots, a_n) > 0$ for all $a_1, \ldots, a_n \in U_{\mathcal{A}}$.