Rewrite Relations

Let E be a set of equations.

The rewrite relation $\rightarrow_E \subseteq \mathrm{T}_\Sigma(X) \times \mathrm{T}_\Sigma(X)$ is defined by

$$
\begin{aligned}
s \rightarrow_E t \quad \text{iff} \quad & \text{there exist } (l \approx r) \in E,\ p \in \mathrm{Pos}(s), \\
& \text{and } \sigma : X \rightarrow \mathrm{T}_\Sigma(X), \\
& \text{such that } s/p = l\sigma \text{ and } t = s[r\sigma]_p.
\end{aligned}
$$

An instance of the lhs (left-hand side) of an equation is called a redex (reducible expression).
Contracting a redex means replacing it with the corresponding instance of the rhs (right-hand side) of the rule.

Rewrite Relations

An equation $l \approx r$ is also called a rewrite rule, if l is not a variable and $\mathrm{Var}(l) \supseteq \mathrm{Var}(r)$.

Notation: $l \to r$.

A set of rewrite rules is called a term rewrite system (TRS).

Rewrite Relations

We say that a set of equations E or a TRS R is terminating, if the rewrite relation $\rightarrow_E$ or $\rightarrow_R$ has this property.

(Analogously for other properties of abstract reduction systems).

Note: If E is terminating, then it is a TRS.

E-Algebras

Let E be a set of closed equations. A Σ-algebra $\mathcal{A}$ is called an E-algebra, if $\mathcal{A} \models \forall\vec{x}(s \approx t)$ for all $\forall\vec{x}(s \approx t) \in E$.

If $E \models \forall\vec{x}(s \approx t)$ (i.e., $\forall\vec{x}(s \approx t)$ is valid in all E-algebras), we write this also as $s \approx_E t$.

Goal:
Use the rewrite relation $\rightarrow_E$ to express the semantic consequence relation syntactically:

$s \approx_E t$ if and only if $s \leftrightarrow_E^* t$.

E-Algebras

Let E be a set of equations over $\mathsf{T}_\Sigma(X)$. The following inference system allows to derive consequences of E:

E-Algebras

$$E \vdash t \approx t \qquad \text{(Reflexivity)}$$

$$\frac{E \vdash t \approx t'}{E \vdash t' \approx t} \qquad \text{(Symmetry)}$$

$$\frac{E \vdash t \approx t' \qquad E \vdash t' \approx t''}{E \vdash t \approx t''} \qquad \text{(Transitivity)}$$

$$\frac{E \vdash t_1 \approx t'_1 \quad \ldots \quad E \vdash t_n \approx t'_n}{E \vdash f(t_1, \ldots, t_n) \approx f(t'_1, \ldots, t'_n)} \qquad \text{(Congruence)}$$

$$E \vdash t\sigma \approx t'\sigma \qquad \text{(Instance)}$$

if $(t \approx t') \in E$ and $\sigma : X \to \mathrm{T}_\Sigma(X)$

E-Algebras

Lemma:

The following properties are equivalent:

(i) $s \leftrightarrow_E^* t$

(ii) $E \vdash s \approx t$ is derivable.

Proof:

(i)$\Rightarrow$(ii): $s \leftrightarrow_E t$ implies $E \vdash s \approx t$ by induction on the depth of the position where the rewrite rule is applied;
then $s \leftrightarrow_E^* t$ implies $E \vdash s \approx t$ by induction on the number of rewrite steps in $s \leftrightarrow_E^* t$.

(ii)$\Rightarrow$(i): By induction on the size of the derivation for $E \vdash s \approx t$.

E-Algebras

Constructing a quotient algebra:

Let X be a set of variables.

For $t \in \mathsf{T}_\Sigma(X)$ let $[t] = \{\, t' \in \mathsf{T}_\Sigma(X) \mid E \vdash t \approx t' \,\}$ be the congruence class of t.

Define a Σ-algebra $\mathsf{T}_\Sigma(X)/E$ (abbreviated by $\mathcal{T}$) as follows:

$U_\mathcal{T} = \{\, [t] \mid t \in \mathsf{T}_\Sigma(X) \,\}$.

$f_\mathcal{T}([t_1], \ldots, [t_n]) = [f(t_1, \ldots, t_n)]$ for $f/n \in \Omega$.

E-Algebras

Lemma:

$f_{\mathcal{T}}$ is well-defined:

If $[t_i] = [t'_i]$, then $[f(t_1, \ldots, t_n)] = [f(t'_1, \ldots, t'_n)]$.

Proof:

Follows directly from the *Congruence* rule for $\vdash$.

E-Algebras

Lemma:

$\mathcal{T} = \mathrm{T}_\Sigma(X)/E$ is an E-algebra.

Proof:

Let $\forall\vec{x}(s \approx t)$ be an equation in E; let α be an arbitrary assignment.

We have to show that $\mathcal{T}(\alpha)(\forall\vec{x}(s \approx t)) = 1$, or equivalently, that $\mathcal{T}(\beta)(s) = \mathcal{T}(\beta)(t)$ for all $\beta = \alpha[\,x_i \mapsto [t_i] \mid i \in I\,]$ with $[t_i] \in U_{\mathcal{T}}$.

Let $\sigma = \{x_i \mapsto t_i \mid i \in I\}$, then $s\sigma \in \mathcal{T}(\beta)(s)$ and $t\sigma \in \mathcal{T}(\beta)(t)$.

By the *Instance* rule, $E \vdash s\sigma \approx t\sigma$ is derivable,
hence $\mathcal{T}(\beta)(s) = [s\sigma] = [t\sigma] = \mathcal{T}(\beta)(t)$.

E-Algebras

Lemma:

Let X be a countably infinite set of variables; let $s, t \in \mathrm{T}_\Sigma(X)$. If $\mathrm{T}_\Sigma(X)/E \models \forall\vec{x}(s \approx t)$, then $E \vdash s \approx t$ is derivable.

Proof:

Assume that $\mathcal{T} \models \forall\vec{x}(s \approx t)$, i.e., $\mathcal{T}(\alpha)(\forall\vec{x}(s \approx t)) = 1$. Consequently, $\mathcal{T}(\beta)(s) = \mathcal{T}(\beta)(t)$ for all $\beta = \alpha[\,x_i \mapsto [t_i] \mid i \in I\,]$ with $[t_i] \in U_{\mathcal{T}}$.

Choose $t_i = x_i$, then $[s] = \mathcal{T}(\beta)(s) = \mathcal{T}(\beta)(t) = [t]$, so $E \vdash s \approx t$ is derivable by definition of $\mathcal{T}$.

E-Algebras

Theorem ("Birkhoff's Theorem"):
Let X be a countably infinite set of variables, let E be a set of (universally quantified) equations. Then the following properties are equivalent for all $s, t \in \mathrm{T}_\Sigma(X)$:

(i) $s \leftrightarrow_E^* t$.

(ii) $E \vdash s \approx t$ is derivable.

(iii) $s \approx_E t$, i.e., $E \models \forall \vec{x}(s \approx t)$.

(iv) $\mathrm{T}_\Sigma(X)/E \models \forall \vec{x}(s \approx t)$.

E-Algebras

Proof:

(i)$\Leftrightarrow$(ii): See above (slide 7).

(ii)$\Rightarrow$(iii): By induction on the size of the derivation for $E \vdash s \approx t$.

(iii)$\Rightarrow$(iv): Obvious, since $\mathcal{T} = \mathcal{T}_E(X)$ is an E-algebra.

(iv)$\Rightarrow$(ii): See above (slide 11).

Universal Algebra

$\mathrm{T}_\Sigma(X)/E = \mathrm{T}_\Sigma(X)/{\approx_E} = \mathrm{T}_\Sigma(X)/{\leftrightarrow_E^*}$ is called the free E-algebra with generating set $X/{\approx_E} = \{\, [x] \mid x \in X \,\}$:

Every mapping $\varphi : X/{\approx_E} \to \mathcal{B}$ for some E-algebra $\mathcal{B}$ can be extended to a homomorphism $\hat{\varphi} : \mathrm{T}_\Sigma(X)/E \to \mathcal{B}$.

$\mathrm{T}_\Sigma(\emptyset)/E = \mathrm{T}_\Sigma(\emptyset)/{\approx_E} = \mathrm{T}_\Sigma(\emptyset)/{\leftrightarrow_E^*}$ is called the initial E-algebra.

Universal Algebra

$\approx_E = \{\, (s, t) \mid E \models s \approx t \,\}$
is called the equational theory of E.

$\approx_E^I = \{\, (s, t) \mid \mathrm{T}_\Sigma(\emptyset)/E \models s \approx t \,\}$
is called the inductive theory of E.

Example:

Let $E = \{\forall x(x + 0 \approx x),\ \forall x \forall y(x + s(y) \approx s(x + y))\}$.
Then $x + y \approx_E^I y + x$, but $x + y \not\approx_E y + x$.

Rewrite Relations

Corollary:
If E is convergent (i.e., terminating and confluent),
then $s \approx_E t$ if and only if $s \leftrightarrow_E^* t$ if and only if $s{\downarrow_E} = t{\downarrow_E}$.

Corollary:
If E is finite and convergent, then $\approx_E$ is decidable.

Reminder:
If E is terminating, then it is confluent if and only if
it is locally confluent.

Rewrite Relations

Problems:

Show local confluence of E.

Show termination of E.

Transform E into an equivalent set of equations that is locally confluent and terminating.

Rewrite Relations

Showing local confluence (Sketch):

Problem: If $t_1 \leftarrow_E t_0 \rightarrow_E t_2$, does there exist a term s such that $t_1 \rightarrow_E^* s \leftarrow_E^* t_2$?

If the two rewrite steps happen in different subtrees (disjoint redexes): yes.

If the two rewrite steps happen below each other (overlap at or below a variable position): yes.

If the left-hand sides of the two rules overlap at a non-variable position: needs further investigation.

Rewrite Relations

Showing local confluence (Sketch):

Question: Are there rewrite rules $l_1 \to r_1$ and $l_2 \to r_2$ such that l_1 and some subterm l_2/p have a common instance $l_1\sigma_1 = (l_2/p)\sigma_2$?

Without loss of generality: assume that the two rewrite rules do not have common variables.
Then: Only a single substitution required: $l_1\sigma = (l_2/p)\sigma$?

Further questions:
For which substitutions σ can this happen?
If there are infinitely many substitutions, can we describe them finitely?