Checking Unsatisfiability

Theorem:
Unsatisfiability of finite sets of first-order formulas (or clauses) is undecidable.

Theorem:
Unsatisfiability of finite sets of first-order formulas (or clauses) is recursively enumerable.

Proposition:
The resolution calculus and the tableaux calculus are sound and refutationally complete semi-decision procedures for unsatisfiability of finite sets of first-order clauses without equality.

Handling Equality Naively

Proposition:

Let F be a closed first-order formula with equality. Let $\sim\, \notin \Omega$ be a new predicate symbol. The set $Eq(\Sigma)$ contains the formulas

$$
\begin{gathered}
\forall x\,(x \sim x)\\
\forall x, y\,(x \sim y \rightarrow y \sim x)\\
\forall x, y, z\,(x \sim y \wedge y \sim z \rightarrow x \sim z)\\
\forall \vec{x}, \vec{y}\,(x_1 \sim y_1 \wedge \cdots \wedge x_n \sim y_n \rightarrow f(x_1, \ldots, x_n) \sim f(y_1, \ldots, y_n))\\
\forall \vec{x}, \vec{y}\,(x_1 \sim y_1 \wedge \cdots \wedge x_n \sim y_n \wedge p(x_1, \ldots, x_n) \rightarrow p(y_1, \ldots, y_n))
\end{gathered}
$$

for every $f/n \in \Omega$ and $p/n \in \Pi$. Let $\tilde{F}$ be the formula that one obtains from F if every occurrence of $\approx$ is replaced by $\sim$. Then F is satisfiable if and only if $Eq(\Sigma) \cup \{\tilde{F}\}$ is satisfiable.

Handling Equality Naively

By giving the equality axioms explicitly, first-order problems with equality can in principle be solved by a standard resolution or tableaux prover.

But this is unfortunately not efficient
(mainly due to the transitivity and congruence axioms).

Roadmap

How to proceed:

1. Arbitrary binary relations.

2. Term rewrite systems.

3. Expressing semantic consequence syntactically.

4. Entailment for equations (unit clauses).

5. Entailment for equational clauses.

2 Abstract Reduction Systems

Abstract Reduction Systems

Abstract reduction system: $(A, \rightarrow)$, where

A is a set,

$\rightarrow \subseteq A \times A$ is a binary relation on A.

Abstract Reduction Systems

$\to^0 = \{ (x,x) \mid x \in A \}$	identity
$\to^{i+1} = \to^i \circ \to$	$i+1$-fold composition
$\to^+ = \bigcup_{i>0} \to^i$	transitive closure
$\to^* = \to^+ \cup \to^0$	reflexive transitive closure
$\to^= = \to \cup \to^0$	reflexive closure
$\to^{-1} = \leftarrow = \{ (x,y) \mid y \to x \}$	inverse
$\leftrightarrow = \to \cup \leftarrow$	symmetric closure
$\leftrightarrow^+ = (\leftrightarrow)^+$	transitive symmetric closure
$\leftrightarrow^* = (\leftrightarrow)^*$	refl. trans. symmetric closure

Abstract Reduction Systems

$x \in A$ is reducible, if there is a y such that $x \to y$.

x is in normal form (irreducible), if it is not reducible.

y is a normal form of x, if $x \to^* y$ and y is in normal form.
Notation: $x{\downarrow}$ (if the normal form of x is uniquely determined).

x and y are joinable, if there is a z such that $x \to^* z \leftarrow^* y$.
Notation: $x \downarrow y$.

Abstract Reduction Systems

A relation $\rightarrow$ is called

Church-Rosser, if $x \leftrightarrow^* y$ implies $x \downarrow y$.

confluent, if $x \leftarrow^* z \rightarrow^* y$ implies $x \downarrow y$.

locally confluent, if $x \leftarrow z \rightarrow y$ implies $x \downarrow y$.

terminating, if there is no infinite decreasing chain $x_0 \rightarrow x_1 \rightarrow x_2 \rightarrow \ldots$.

normalizing, if every $x \in A$ has a normal form.

convergent, if it is confluent and terminating.

Abstract Reduction Systems

Lemma:

If $\rightarrow$ is terminating, then it is normalizing.

Note: The reverse implication does not hold.

Abstract Reduction Systems

Theorem:

The following properties are equivalent:

(i) $\quad \rightarrow$ has the Church-Rosser property.

(ii) $\quad \rightarrow$ is confluent.

Proof:

(i)$\Rightarrow$(ii): trivial.

(ii)$\Rightarrow$(i): by induction on the number of peaks in the derivation $x \leftrightarrow^* y$.

Abstract Reduction Systems

Lemma:
If $\rightarrow$ is confluent, then every element has at most one normal form.

Corollary:
If $\rightarrow$ is normalizing and confluent, then every element x has a unique normal form.

Theorem:
If $\rightarrow$ is normalizing and confluent, then $x \leftrightarrow^* y$ if and only if $x{\downarrow} = y{\downarrow}$.

Well-Founded Orderings

A (strict) partial ordering $(A, >)$ is a transitive and irreflexive binary relation on A.

A (strict) partial ordering $>$ is called well-founded (Noetherian), if there is no infinite decreasing chain $x_0 > x_1 > x_2 > \ldots$.

Well-Founded Orderings

Lemma:

If $\rightarrow$ is a terminating binary relation over A,
then $\rightarrow^+$ is a well-founded partial ordering.

Lemma:

If $>$ is a well-founded partial ordering and $\rightarrow \subseteq >$,
then $\rightarrow$ is terminating.

Well-Founded Orderings

Proposition:
$(A, >)$ is well-founded, if and only if every non-empty subset of A has a minimal element.

Proof:
If $(A, >)$ is not well-founded, then there exists an infinite decreasing chain $x_0 > x_1 > x_2 > \dots$.
Then $\{ x_i \mid i \in \mathbb{N} \}$ does not have a minimal element.

Conversely, if there exists a non-empty subset $B \subseteq A$ without minimal element, then for every $x \in B$ there exists a smaller $y \in B$.
Hence there is an infinite decreasing chain of elements of B.

Well-Founded Orderings

Theorem ("Well-founded induction principle"):
Let $(A, >)$ be a well-founded partial ordering;
let P be a unary predicate on A.

If for all $x \in A$ the following property holds:

$$\text{if } P(y) \text{ for all } y < x, \text{ then } P(x) \qquad (*)$$

then $P(x)$ for all $x \in A$.

Proof:
Assume that $B = \{\, x \in A \mid \neg P(x) \,\}$ is not empty.
Since $>$ is well-founded, B has a minimal element.
This element violates $(*)$.

Proving Confluence

Lemma ("Newman's Lemma"):
If a terminating relation $\rightarrow$ is locally confluent, then it is confluent.

Proof:
Let $\rightarrow$ be a terminating and locally confluent relation.
Then $\rightarrow^{+}$ is a well-founded ordering.
Define $P(z) \;\Leftrightarrow\; \big(\forall x, y : x \leftarrow^{*} z \rightarrow^{*} y \Rightarrow x \downarrow y\big)$.
Prove $P(z)$ for all $x \in A$ by well-founded induction over $\rightarrow^{+}$:
Case 1: $x \leftarrow^{0} z \rightarrow^{*} y$: trivial.
Case 2: $x \leftarrow^{*} z \rightarrow^{0} y$: trivial.
Case 3: $x \leftarrow^{*} x' \leftarrow z \rightarrow y' \rightarrow^{*} y$: use local confluence, then use the induction hypothesis.