Problem 1 *(Abstract Reduction Systems)* (5 points)

Prove: There is no abstract reduction system $(A, \rightarrow)$ with $A \neq \emptyset$ such that every $a \in A$ has at least two normal forms.

Problem 2 *(E-Algebras)* ($4 + 4 = 8$ points)

Let $\Sigma = (\Omega, \emptyset)$ with $\Omega = \{a/0,\ b/0,\ c/0,\ f/1\}$; let E be the set of (implicitly universally quantified) equations $\{\, a \approx f(b),\ f(f(x)) \approx f(f(y)) \,\}$.

Part (a)

Give one possible derivation for the statement $E \vdash f(a) \approx f(f(f(z)))$.

Part (b)

Is the universe of the initial E-algebra $\mathrm{T}_\Sigma(\emptyset)/E$ finite or infinite? If it is finite, how many elements does it have?

Problem 3 *(Rewrite Orderings)* (6 points)

Let $\mathcal{A}$ be a Σ-algebra; let $>_1$ be a strict partial ordering on its universe such that the interpretation $f_\mathcal{A}$ of every function symbol f is monotone w.r.t. $>_1$. Let $>_2$ be a strict partial ordering on $\mathrm{T}_\Sigma(X)$ that is compatible with Σ-operations.
Define the ordering $>_{12}$ over $\mathrm{T}_\Sigma(X)$ by

$$\begin{array}{lll} s >_{12} t & \text{iff} & \mathcal{A}(\alpha)(s) >_1 \mathcal{A}(\alpha)(t) \text{ for all } \alpha : X \rightarrow U_\mathcal{A} \\ & & \text{or } \mathcal{A}(\alpha)(s) \geq_1 \mathcal{A}(\alpha)(t) \text{ for all } \alpha : X \rightarrow U_\mathcal{A} \text{ and } s >_2 t. \end{array}$$

Show that $>_{12}$ is compatible with Σ-operations.

Problem 4 *(Lexicographic Path Orderings)* (6 points)

Let $\Sigma = (\Omega, \emptyset)$ with $\Omega = \{f/1,\ g/1,\ h/2,\ k/1\}$; let R be the following rewrite system:

$$\begin{aligned} h(y, f(x)) &\rightarrow h(k(x), x), \\ k(g(x)) &\rightarrow f(x), \\ k(k(x)) &\rightarrow h(x, x) \end{aligned}$$

Find a precedence $>$ on Ω such that $\rightarrow_R \subseteq >_{\mathrm{lpo}}$, where the function symbol h has left-to-right lexicographic status.

Problem 5 *(Critical Pairs)* (7 points)

Let R be the following rewrite system:

$$\begin{aligned} g(f(x,x)) &\to g(x), \\ f(g(y),g(a)) &\to b, \\ a &\to f(b,b) \end{aligned}$$

Compute all critical pairs between rules in R and check whether they are joinable in R.

Problem 6 *(Superposition Calculus)* (6 points)

Give an example of a signature Σ, a reduction ordering $>$ that is total on ground Σ-terms, and a set N of two ground clauses such that

- $R_\infty \neq \emptyset$,
- all clauses in N are true in R_∞, and
- N is not saturated up to redundancy.