Problem 1 *(Unification)* (6 points)

For each of the following unification problems, compute either an mgu or show that it is not unifiable:

$$E_1 = \{f(g(x), x) =^? f(y, h(y))\}$$
$$E_2 = \{h(a, z, z, b) =^? h(x, x, y, y)\}$$
$$E_3 = \{g(x, f(x)) =^? g(y, z),\ g(x', x') =^? g(y, f(z'))\}$$

Problem 2 *(Well-founded Orderings)* (6 points)

Let $(A, >)$ be a well-founded partial ordering, let $f : A \to A$ be a monotone function (that is, $x > y$ implies $f(x) > f(y)$ for all elements $x, y \in A$). Prove: If $x \geq f(x)$ for all $x \in A$, then $x = f(x)$ for all $x \in A$.

Problem 3 *(Reduction Orderings)* (8 points)

The proper subterm relation $\rhd$ is defined by

$s \rhd t$ if and only if there is a $p \in \mathrm{Pos}(s)$ such that $p \neq \varepsilon$ and $s/p = t$.

Is the proper subterm relation a reduction ordering? Give a proof or a counterexample.

Problem 4 *(Multisets)* (4 + 4 = 8 points)

Let $N = \{M_1, M_2, M_3, M_4, M_5\}$ be a set of multisets of multisets:

$$M_1 = \{\{a_4\}, \{a_4\}, \{a_1\}, \{a_1\}\}$$
$$M_2 = \{\{a_2\}, \{a_1\}, \{a_1\}\}$$
$$M_3 = \{\{a_3, a_1\}\}$$
$$M_4 = \{\{a_4, a_3\}, \{a_3, a_2\}, \{a_2, a_1, a_1\}\}$$
$$M_5 = \{\{a_2\}, \{a_1, a_1\}, \emptyset\}$$

Part (a)

Let the ordering $\succ$ be defined by $a_4 \succ a_3 \succ a_2 \succ a_1$, let $\succ_m$ be the multiset extension of $\succ$, and let $\succ_{mm}$ be the multiset extension of $\succ_m$. Sort the elements of N with respect to $\succ_{mm}$.

Part (b)

Find another total ordering $\succ'$ on $\{a_1, a_2, a_3, a_4\}$ such that M_3 is maximal and M_1 is minimal in N with respect to $\succ'_{mm}$, where $\succ'_{mm}$ is the twofold multiset extension of $\succ'$.

Problem 5 *(Confluence)* (10 points)

For a term rewrite system R, we define $\mathrm{LSymb}(R)$ as the set of all function symbols occurring in the left-hand sides of rules in R. More formally,

$$\mathrm{LSymb}(R) = \bigcup_{l \to r \in R} \mathrm{Symb}(l),$$

where $\mathrm{Symb}(x) = \emptyset$ and $\mathrm{Symb}(f(t_1, \ldots, t_n)) = \{f\} \cup \bigcup_{i=1}^{n} \mathrm{Symb}(t_i)$.
Prove: If R_1 and R_2 are confluent term rewrite systems, such that $R_1 \cup R_2$ is terminating, and $\mathrm{LSymb}(R_1) \cap \mathrm{LSymb}(R_2) = \emptyset$, then $R_1 \cup R_2$ is confluent.