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Tutorials for "Automated Reasoning"
Exercise sheet 2

Exercise 2.1:
(a) Prove: If $>$ is a well-founded strict partial ordering on a set M and if b is the only element of M that is minimal in M, then b is the smallest element of M.

(b) Give an example of a strict partial ordering $>$ on a set M and an element $b \in M$ such that b is the only element of M that is minimal in M but not the smallest element of M.

Exercise 2.2:
For an alphabet Σ with a well-founded ordering $>_\Sigma$ let the relation $>_{\Sigma,\mathrm{lex}} \subseteq \Sigma^* \times \Sigma^*$ be defined by $w >_{\Sigma,\mathrm{lex}} w'$ if and only if w and w' have the same length n and w is larger than w' in the n-fold lexicographic combination of $>_\Sigma$. Let the relation $\succ$ be defined as

$$w \succ w' :\Leftrightarrow |w| > |w'| \text{ or } (|w| = |w'| \text{ and } w >_{\Sigma,\mathrm{lex}} w').$$

Prove that $\succ$ is a well-founded ordering on Σ^*. (Note: We define the 0-fold lexicographic combination of an ordering as $\emptyset$ and the 1-fold lexicographic combination of an ordering as the ordering itself. You may use the fact that for any $n \in \mathbb{N}$ the n-fold lexicographic combination of a well-founded ordering is well-founded.)

Exercise 2.3:
Let M be the set $\{a, b, c\}$ and let the ordering $\succ$ be defined by $a \succ b$ and $a \succ c$. (b and c are incomparable!) Consider the following multisets over multisets over M:

(1) $\{\{a, a\}\}$

(2) $\{\{a, b, c\}, \{b, c\}\}$

(3) $\{\{a, b, c\}, \{b, b\}, \{c, c\}\}$

(4) $\{\{a, c\}, \{b, b, c, c\}\}$

Determine for each pair of multisets whether they are comparable with respect to $(\succ_{\mathrm{mul}})_{\mathrm{mul}}$, and, if so, which multiset is larger.

Exercise 2.4:
Determine all strict total orderings $\succ$ on the set $\{a, b, c, d, e\}$ such that the following properties hold simultaneously:

(1) $\{a, b\} \succ_{\text{mul}} \{a, a, c\}$

(2) $\{c, d\} \succ_{\text{mul}} \{b, b, b\}$

(3) $\{a, e\} \succ_{\text{mul}} \{c, e, e\}$

Exercise 2.5:
Let M be a set, let $\succ$ be a strict partial ordering over M. Let $b, b_1, b_2 \in M$ and let S, S_1, S_2 be finite multisets over M.

(a) Prove or refute: If $\{b\} \succ_{\text{mul}} S_1$ and $\{b\} \succ_{\text{mul}} S_2$, then $\{b\} \succ_{\text{mul}} S_1 \cup S_2$.

(b) Prove or refute: If $S \succ_{\text{mul}} \{b_1\}$ and $S \succ_{\text{mul}} \{b_2\}$, then $S \succ_{\text{mul}} \{b_1, b_2\}$.

Bring your solution to the tutorial on November 4 and compare it with the solution that is discussed there. If you are still unsure afterwards whether your solution is correct or not, feel free to ask the instructor after the tutorial. Your solution will not be graded.