MAX PLANCK INSTITUTE FOR INFORMATICS

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Tutorials for "Automated Reasoning"
Exercise sheet 11

Exercise 11.1:
Let N be the following set of ground clauses:

$$\neg P_3 \vee P_1 \vee P_1 \qquad (1)$$
$$\neg P_2 \vee P_1 \qquad (2)$$
$$P_4 \vee P_4 \qquad (3)$$
$$P_3 \vee \neg P_2 \qquad (4)$$
$$P_4 \vee P_3 \qquad (5)$$

(a) Find a total atom ordering $\succ$ such that both clause (2) and (5) are redundant w.r.t. N.

(b) Prove that there is no atom ordering such that clause (4) is redundant w.r.t. N.

Exercise 11.2:
Prove the details of Thm. 3.49 (ii):

(1) Prove that $M \subseteq \mathit{Red}(N) \Rightarrow \mathit{Red}(N) \subseteq \mathit{Red}(N \setminus M)$ for ground clause sets M and N.

(2) Prove that for a general clause set N, every clause in $G_\Sigma(N) \setminus \mathit{Red}(G_\Sigma(N))$ is an instance of a clause in $N \setminus \mathit{Red}(N)$.

(3) Prove that $M \subseteq \mathit{Red}(N) \Rightarrow \mathit{Red}(N) \subseteq \mathit{Red}(N \setminus M)$ for general clause sets M and N.

Exercise 11.3:
Prove Lemma 3.50: Let $N_0 \vdash N_1 \vdash N_2 \vdash \ldots$ be a run. Then $\mathit{Red}(N_i) \subseteq \mathit{Red}(\bigcup_{i\geq 0} N_i)$ and $\mathit{Red}(N_i) \subseteq \mathit{Red}(N_\infty)$ for every i.

Exercise 11.4:
Let $\Sigma = (\Omega, \emptyset)$ with $\Omega = \{b/0,\ f/1,\ g/1\}$; let E be the set of (implicitly universally quantified) equations $\{\, f(g(f(x))) \approx b \,\}$.

Give one possible derivation for the statement $E \vdash f(g(b)) \approx b$.

Exercise 11.5:
Let $\Sigma = (\Omega, \emptyset)$ with $\Omega = \{f/1,\ b/0,\ c/0,\ d/0\}$; let E be the set of (implicitly universally quantified) equations $\{\, f(f(x)) \approx b \,\}$.

(a) Show that $b \leftrightarrow_E^* f(b)$. How does the rewrite proof look?

(b) Is the universe of the initial E-algebra $\mathrm{T}_\Sigma(\emptyset)/E$ finite or infinite? If it is finite, how many elements does it have?

Exercise 11.6:
Challenge Problem:

(1) Show that the compactness theorem (Thm. 3.42) holds also for first-order logic with equality. (You may use all results proved in the lecture so far.)

(2) Use the compactness theorem for first-order logic with equality to prove the following statement: Let F be a first-order formula with equality. If, for every natural number n, F has a model whose universe has at least n elements, then F has a model with an infinite universe.

Bring your solution to the tutorial on January 20 and compare it with the solution that is discussed there. If you are still unsure afterwards whether your solution is correct or not, feel free to ask the instructor after the tutorial. Your solution will not be graded.