Universität
des
Saarlandes
FR Informatik

Uwe Waldmann **October 17, 2025**

Tutorials for "Automated Reasoning"
Exercise sheet 1

Exercise 1.1:
Find an abstract reduction system $(A, \to)$, such that the relations $\to$, $\leftrightarrow$, $\leftrightarrow^+$, and $\leftrightarrow^*$ are all different.

Exercise 1.2:
Find an abstract reduction system $(B, \to)$, such that $\to^+$ is irreflexive and $\to$ is normalizing, but not terminating.

Exercise 1.3:
Let $(\mathbb{Q}, >)$ be the set of rational numbers with the usual ordering $>$. Construct infinite subsets M_1, M_2, M_3, and M_4 of $\mathbb{Q}$ with the following properties

(1) $>$ is well-founded on M_1 and M_1 has a minimal element.

(2) $>$ is not well-founded on M_2 and M_2 has a minimal element.

(3) $>$ is well-founded on M_3 and M_3 does not have a maximal element.

(4) $>$ is well-founded on M_4 and M_4 has a maximal element.

Exercise 1.4:
Let $(M, \succ)$ be an ordering and $b, c \in M$. We say that b is a *successor* of c, if $b \succ c$ and if there exists no $d \in M$ with $b \succ d$ and $d \succ c$.

(1) Prove: If $\succ$ is well-founded, then every element of M has either a successor or it is maximal.

(2) Prove: If $\succ$ is well-founded and total, then every element of M has at most one successor.

(3) Give an example of a set M, a well-founded ordering $\succ$ on M, and an element $b \in M$ such that b is neither a minimal element of M nor a successor of any other element of M.

Exercise 1.5:
You are asked to review a scientific article that has been submitted to a conference on automated reasoning. On page 3 of the article, the authors write the following:

> **Theorem 2.** Let $\to_1$ and $\to_2$ be two binary relations over a non-empty set M. If $\to_1$ and $\to_2$ are terminating, then $\to_1 \cup \to_2$ is also terminating.
>
> **Proof.** Since $\to_1$ is terminating, $\to_1^+$ is a well-founded ordering. Assume that there exists an infinite descending $(\to_1 \cup \to_2)$-chain. Since $\to_1^+$ is well-founded, there exists a minimal element b with respect to $\to_1^+$ such that there is an infinite descending $(\to_1 \cup \to_2)$-chain starting with b.
>
> Case 1: The $(\to_1 \cup \to_2)$-chain starts with a $\to_1$-step $b \to_1 b'$. The rest of the chain, starting with b', is still infinite. However, b' is smaller than b with respect to $\to_1^+$. This contradicts the minimality of b.
>
> Case 2: The $(\to_1 \cup \to_2)$-chain starts with a $\to_2$-step $b \to_2 b'$. Since $\to_2$ is terminating, the chain cannot consist only of $\to_2$-steps. Therefore there must be some $\to_1$-step in the chain, say $b'' \to_1 b'''$. Hence there exists an infinite $(\to_1 \cup \to_2)$-chain starting with this step. But as we have seen in Case 1, an infinite $(\to_1 \cup \to_2)$-chain cannot start with a $\to_1$-step. So there is again a contradiction.
>
> Consequently, every descending $(\to_1 \cup \to_2)$-chain must be finite, which means that $\to_1 \cup \to_2$ is terminating.

(1) Is the "proof" correct (yes/no)?

(2) If the "proof" is not correct:

  (a) Which steps are incorrect?

  (b) Does the "theorem" hold? If yes, give a correct proof, otherwise give a counterexample.

Bring your solution to the tutorial on October 28 and compare it with the solution that is discussed there. If you are still unsure afterwards whether your solution is correct or not, feel free to ask the instructor after the tutorial. Your solution will not be graded.