Automated Reasoning I, 2021 Midterm Exam, Sample Solution

Assignment 1

Part (a) Let (V, E) be given, let $C = \{0, 1, 2\}$ be the set of "colors". Let $\Pi = \{ P_v^c \mid v \in V, c \in C \}$, where P_v^c is supposed to be true in a model if and only if $\phi(v) = c$. Then N is the following set of clauses over Π:

- $\bigvee_{c \in C} P_v^c$ for every $v \in V$ (that is, v is mapped to some $c \in C$ by ϕ).
- $\neg P_v^c \vee \neg P_v^{c'}$ for every $v \in V$ and all $c, c' \in C$ with $c < c'$ (that is, v is not mapped to both c and c').
- $\neg P_v^c \vee \neg P_{v'}^c$ for every edge $(v, v') \in E$ and every $c \in C$ (that is, v and v' are not both mapped to c).

Part (b) There are several possible translations. We can for instance extend Π and N from Part (a) in the following way: Let $\Pi' = \Pi \cup \{ Q_{v,v'}^{c,c'} \mid (v, v') \in E, c, c' \in C, c < c' \}$, where the propositional variable $Q_{v,v'}^{c,c'}$ is supposed to be true in a model only if $\phi(v) = c$ and $\phi(v') = c'$ or $\phi(v) = c'$ and $\phi(v') = c$. Then N' adds the following clauses to N:

- $\bigvee_{(v,v') \in E} Q_{v,v'}^{c,c'}$ for all $c, c' \in C$ with $c < c'$ (that is, at least one edge connects two vertices with colors c and c').
- $\neg Q_{v,v'}^{c,c'} \vee P_v^c \vee P_{v'}^c$ for every edge $(v, v') \in E$ and all $c, c' \in C$ with $c < c'$ (that is, one of v and v' is mapped to c).
- $\neg Q_{v,v'}^{c,c'} \vee P_v^{c'} \vee P_{v'}^{c'}$ for every edge $(v, v') \in E$ and all $c, c' \in C$ with $c < c'$ (that is, one of v and v' is mapped to c').

Assignment 2

Part (a) Since clause (6) is a conflict clause and contains the complement of the deduced literal S, we resolve (6) with the clause used to propagate S, namely (5), and obtain $T \vee \neg U \vee \neg W$ (which is not a backjump clause). By resolving this clause with the clause used to propagate $\neg T$, namely (4), we obtain $P \vee \neg U \vee \neg W$ (10) (which is a backjump clause). The best possible successor state for this backjump clause is $\neg P^{\mathrm{d}}\ W\ \neg U \parallel N \cup \{(10)\}$.

Grading scheme: 4 points for computing the 1UIP backjump clause; 3 points for determining the optimal successor state.

Part (b) Clause (9) is an asymmetric tautology w. r. t. $N \setminus \{(9)\}$; therefore is has the RAT property and may be deleted. To see that we add the negation of (9), that is, the three unit clauses $\neg S$ (11), U (12), and $\neg V$ (13) to N and try to derive a contradiction by unit propagation. (Note that we may not use (9) itself for unit propagation.) We obtain

$$\underset{(11)}{\neg S}\quad \underset{(12)}{U}\quad \underset{(13)}{\neg V}\quad \underset{(2)}{Q}\quad \underset{(8)}{\neg P}\quad \underset{(1)}{W}\quad \parallel N \setminus \{(9)\} \cup \{(11), (12), (13)\}$$

At this point, (5) is a conflict clause, so we have shown that $N \setminus \{(9)\} \cup \{\neg\,(9)\} \models \bot$ and therefore $N \setminus \{(9)\} \models \{(9)\}$.

Assignment 3

Assume that $\mathcal{A} \not\models F$ and $\mathcal{A} \not\models C$ and that every propositional variable that occurs in F occurs also in C. We have to show that $\mathcal{B} \models F$ implies $\mathcal{B} \models C$ for every valuation $\mathcal{B}$: Suppose that $\mathcal{B} \models F$. Then there must exist a propositional variable P that occurs in F and for which $\mathcal{A}(P) \neq \mathcal{B}(P)$. By assumption, the propositional variable P occurs also in C. Now there are two possibilities: Either $\mathcal{A}(P) = 1$, then $\mathcal{A} \not\models C$ implies that C contains the negative literal $\neg P$, and since $\mathcal{B}(P) = 0$ we have $\mathcal{B}(C) = 1$. Otherwise $\mathcal{A}(P) = 0$, then $\mathcal{A} \not\models C$ implies that C contains the positive literal P, and since $\mathcal{B}(P) = 1$ we have again $\mathcal{B}(C) = 1$.

Assignment 4

Part (a) Assume that the Σ-formula F is valid. Let $\mathcal{A}$ and β be an arbitrary Σ-algebra and an assignment. We have to show that $\mathcal{A}(\beta)(\mathrm{rep}(F)) = 1$. Define a Σ-algebra $\mathcal{B}$ such that $U_{\mathcal{B}} = U_{\mathcal{A}}$, $f_{\mathcal{B}} = f_{\mathcal{A}}$ for every $f \in \Omega$, $Q_{\mathcal{B}} = R_{\mathcal{A}}$, and $P_{\mathcal{B}} = P_{\mathcal{A}}$ for every $P \in \Pi \setminus \{Q\}$. Obviously, $\mathcal{B}(\gamma)(t) = \mathcal{A}(\gamma)(t)$ for every assignment γ and Σ-term t. We show that $\mathcal{B}(\gamma)(G) =$

$\mathcal{A}(\gamma)(\text{rep}(G))$ for every Σ-formula G and every γ by induction over the formula structure:

If $G = Q(s_1, \dots, s_n)$, then $\text{rep}(G) = R(s_1, \dots, s_n)$. The tuple $(\mathcal{A}(\gamma)(s_1), \dots, \mathcal{A}(\gamma)(s_n)) = (\mathcal{B}(\gamma)(s_1), \dots, \mathcal{B}(\gamma)(s_n))$ is contained in $Q_\mathcal{B}$ iff it is contained in $R_\mathcal{A}$ by definition of $Q_\mathcal{B}$, therefore we get $\mathcal{B}(\gamma)(Q(s_1, \dots, s_n)) = \mathcal{A}(\gamma)(R(s_1, \dots, s_n)) = \mathcal{A}(\gamma)(\text{rep}(Q(s_1, \dots, s_n)))$.

If $G = P(t_1, \dots, t_m)$ for some $P \neq Q$, then $\text{rep}(G) = P(s_1, \dots, s_n)$. The tuple $(\mathcal{A}(\gamma)(s_1), \dots, \mathcal{A}(\gamma)(s_n)) = (\mathcal{B}(\gamma)(s_1), \dots, \mathcal{B}(\gamma)(s_n))$ is contained in $P_\mathcal{B}$ iff it is contained in $P_\mathcal{A}$, therefore we get $\mathcal{B}(\gamma)(P(s_1, \dots, s_n)) = \mathcal{A}(\gamma)(\text{rep}(P(s_1, \dots, s_n)))$.

If $G = G' \vee G''$, then $\text{rep}(G) = \text{rep}(G') \vee \text{rep}(G'')$. By induction, $\mathcal{B}(\gamma)(G') = \mathcal{A}(\gamma)(\text{rep}(G'))$ and $\mathcal{B}(\gamma)(G'') = \mathcal{A}(\gamma)(\text{rep}(G''))$, therefore $\mathcal{B}(\gamma)(G) = \mathcal{B}(\gamma)(G' \vee G'') = \max\{\mathcal{B}(\gamma)(G'), \mathcal{B}(\gamma)(G'')\} = \max\{\mathcal{A}(\gamma)(\text{rep}(G')), \mathcal{A}(\gamma)(\text{rep}(G''))\} = \mathcal{A}(\gamma)(\text{rep}(G') \vee \text{rep}(G'')) = \mathcal{A}(\gamma)(\text{rep}(G))$.

If $G = \neg G'$, then $\text{rep}(G) = \neg\text{rep}(G')$. By induction, $\mathcal{B}(\gamma)(G') = \mathcal{A}(\gamma)(\text{rep}(G'))$, therefore $\mathcal{B}(\gamma)(G) = \mathcal{B}(\gamma)(\neg G') = 1 - \mathcal{B}(\gamma)(G') = 1 - \mathcal{A}(\gamma)(\text{rep}(G')) = \mathcal{A}(\gamma)(\neg\text{rep}(G')) = \mathcal{A}(\gamma)(\text{rep}(G))$.

The other cases are handled analogously.

Since F is supposed to be valid, we have therefore $\mathcal{A}(\beta)(\text{rep}(F)) = \mathcal{B}(\beta)(F) = 1$.

Part (b) Let $F = Q(b) \wedge \neg R(b)$, then $\text{rep}(F) = R(b) \wedge \neg R(b)$. Clearly, F is satisfiable, but $\text{rep}(F)$ is unsatisfiable.

Assignment 5

The NNF transformation of

$$\exists w\, \forall x\, \exists z\, \neg\exists y\, \forall v \Big(\neg P(c, v, f(x), y) \wedge \big(Q(v, z) \rightarrow R(x, z, w)\big)\Big)$$

yields

$$\exists w\, \forall x\, \exists z\, \forall y\, \exists v \Big(P(c, v, f(x), y) \vee \big(Q(v, z) \wedge \neg R(x, z, w)\big)\Big)$$

Miniscoping proceeds bottom-up. First, we move $\exists v$ inside the disjunction and then inside the conjunction. Second, we move $\forall y$ inside the disjunction. Third, we move $\exists z$ inside the disjunction:

$$\exists w\, \forall x \Big(\forall y\, \exists v\, P(c, v, f(x), y) \vee \exists z \big(\exists v\, Q(v, z) \wedge \neg R(x, z, w)\big)\Big)$$

At this point, none of the miniscoping rules is applicable anymore. Variable renaming yields

$$\exists w\, \forall x \Big(\forall y\, \exists v\, P(c, v, f(x), y) \vee \exists z \big(\exists v'\, Q(v', z) \wedge \neg R(x, z, w)\big)\Big)$$

Skolemization starts with the *outermost* existential quantifier. First, w is replaced by a *new* constant b. We obtain

$$\forall x \Big(\forall y\, \exists v\, P(c, v, f(x), y) \vee \exists z \big(\exists v'\, Q(v', z) \wedge \neg R(x, z, b)\big)\Big)$$

Then v and z are replaced by *new* functions g (applied to the free variables x and y) and g' (applied to the free variable x), and then v' is replaced by a *new* function g'' (applied to the free variable x). We get

$$\forall x \Big(\forall y\, P(c, g(x, y), f(x), y) \vee \big(Q(g''(x), g'(x)) \wedge \neg R(x, g'(x), b)\big)\Big)$$

The universal quantifiers are pushed upward:

$$\forall x \forall y \Big(P(c, g(x, y), f(x), y) \vee \big(Q(g''(x), g'(x)) \wedge \neg R(x, g'(x), b)\big)\Big)$$

Using the distributivity law, we get the CNF

$$\forall x \forall y \Big(\big(P(c, g(x, y), f(x), y) \vee Q(g''(x), g'(x))\big) \wedge \big(P(c, g(x, y), f(x), y) \vee \neg R(x, g'(x), b)\big)\Big)$$

Grading scheme: 5 points for miniscoping; 5 points for Skolemization; 4 points for the rest.

Assignment 6

Ineq. (1) holds if and only if $P \succ Q$. Ineq. (2) holds if and only if $R \succ P$ or $Q \succ P$, but the second of the two possibilities is excluded by (1). Ineq. (3) holds if and only if $R \succ S$. There are three strict orderings that satisfy these conditions, namely $R \succ S \succ P \succ Q$, $R \succ P \succ S \succ Q$, and $R \succ P \succ Q \succ S$.

Grading scheme: -4 points for each missing or wrong ordering.