# Automated Reasoning I, 2019
# Midterm Exam, Sample Solution

## Assignment 1

Ineq. (1) holds if and only if $b \succ a$ and $b \succ c$. Ineq. (2) holds if and only if $c \succ b$ or $d \succ b$, but the first of the two possibilities is excluded by (1). Finally ineq. (3) holds if and only if $a \succ c$ and $a \succ e$. There are two strict orderings that satisfy these conditions, namely $d \succ b \succ a \succ c \succ e$ and $d \succ b \succ a \succ e \succ c$.

*Grading scheme:* 5 points for each ordering.

## Assignment 2

The property holds. We show by induction that for every $j \in \{0, \dots, n\}$ there is a partial valuation $\mathcal{A}_j$ that satisfies $C_1, \dots, C_j$ and in which exactly $j$ atoms are defined.

If $j = 0$ the statement is trivial: define $\mathcal{A}_0$ as the valuation that is undefined for every propositional variable. If $0 < j \leq n$, we assume by induction that the statement holds for $j - 1$; so there exists a partial valuation $\mathcal{A}_{j-1}$ that satisfies $C_1, \dots, C_{j-1}$ and in which exactly $j - 1$ atoms are defined. As $C_j$ contains $j$ literals that are different and non-complementary, $C_j$ must contain $j$ different atoms. Since only $j - 1$ atoms are defined in $\mathcal{A}_{j-1}$, there exists at least one atom $P$ in $C_j$ that is undefined in $\mathcal{A}_{j-1}$. Now define $\mathcal{A}_j$ as the valuation that maps $P$ to 1 if $P$ occurs positively in $C_j$, or to 0 if $P$ occurs negatively in $C_j$, and that interprets every other atom $Q$ in the same way as $\mathcal{A}_{j-1}$. Since all atoms that are defined in $\mathcal{A}_{j-1}$ are defined in the same way in $\mathcal{A}_j$, $\mathcal{A}_j$ satisfies $C_1, \dots, C_{j-1}$; moreover $\mathcal{A}$ satisfies $C_j$ since it interprets $P$ appropriately.

## Assignment 3

**Part (a)** Since clause (6) is a conflict clause and contains the deduced literal $\neg S$, we resolve (6) with the clause used to propagate $\neg S$, namely (5), and obtain $\neg U \vee \neg V \vee W$ (which is not a backjump clause). By resolving this clause with (4), we obtain $\neg R \vee \neg V \vee W$ (which is not a backjump clause either). By resolving this clause with (2), we obtain $\neg R \vee W$ (7), which is a backjump clause. The best possible successor state for this backjump clause is $P^{\mathrm{d}}\ \neg W\ \neg R \parallel N$.

*Grading scheme:* 5 points for computing the backjump clause according to the 1UIP strategy; 3 points for determining the optimal successor state.

**Part (b)** We obtain an alternative backjump clause if we continue the resolution process, that is, if we resolve (7) and (1). The resolvent is $\neg P \vee \neg R$. Since this clause consists only of complements of decision literals, it is a backjump clause (as indicated in the paragraph "Getting Better Backjump Clauses"). The best possible successor state for this backjump clause is again $P^{\mathrm{d}}\ \neg W\ \neg R \parallel N$.

We obtain a third backjump clause by taking the disjunction of the complements of *all* decision literals on the trail, that is, $\neg P \vee \neg Q \vee \neg R$ (as indicated in the proof of Lemma 2.18). The best possible successor state for this backjump clause is $P^{\mathrm{d}}\ \neg W\ \neg Q^{\mathrm{d}}\ \neg R \parallel N$.

*Grading scheme:* 1 point for each backjump clause; 1 point for each optimal successor state.

## Assignment 4

**Part (a)** Let $\succ$ be a well-founded and total ordering on a set $M$, let $\phi : M^n \to M$ be a function that is strictly monotonic in the $j$-th argument, where $1 \leq j \leq n$. Let $a_1, \dots, a_{j-1}, a_{j+1}, \dots, a_n$ be elements of $M$. We show $\phi(a_1, \dots, a_j, \dots, a_n) \succeq a_j$ for all $a_j \in M$ by well-founded induction over $a_j$ and $\succ$.

Let $b := \phi(a_1, \dots, a_j, \dots, a_n)$. Assume that $b \not\succeq a_j$. Since $\succ$ is total, we conclude that $a_j \succ b$. So by the induction hypothesis, we must have $\phi(a_1, \dots, b, \dots, a_n) \succeq b$. But this implies $\phi(a_1, \dots, a_j, \dots, a_n) = b \preceq \phi(a_1, \dots, b, \dots, a_n)$, contradicting the strict monotonicity of $\phi$ in the $j$-th argument. So $\phi(a_1, \dots, a_j, \dots, a_n) = b \succeq a_j$ as required.

**Part (b)** Let $M = \{b, c\}$, let $\succ = \emptyset$, that is, the ordering in which all elements are incomparable. Now define $\phi(b) = c$ and $\phi(c) = b$. Then

$\phi$ is trivially strictly monotonic in the first argument (since the condition $a_1 \succ a_1'$ is never satisfied), but $\phi(b) \succeq b$ does not hold.

**Part (c)** We use induction over the structure of terms.

If $t$ is a variable $y$, then $x \in \mathrm{var}(t)$ implies $x = y$, so $\mathcal{A}(\beta)(y) = \beta(y)$ by definition of $\mathcal{A}(\beta)$.

If $t$ is a term $f(t_1, \dots, t_n)$, then $x \in \mathrm{var}(t)$ implies $x \in \mathrm{var}(t_i)$ for some $i$. So $\mathcal{A}(\beta)(t) = f_{\mathcal{A}}(\mathcal{A}(\beta)(t_1), \dots, \mathcal{A}(\beta)(t_n)) \succeq \mathcal{A}(\beta)(t_i)$ by strict monotonicity of $f_{\mathcal{A}}$ and part (a), and $\mathcal{A}(\beta)(t_i) \succeq \beta(x)$ by induction for $t_i$.

## Assignment 5

First we convert the entailment problem into an unsatisfiability problem. We know that $N \models F$ if and only if $N \cup \{\neg F\} \models \bot$. So we consider the set of formulas $\big\{P(b) \to Q,\ P(b) \to R,\ P(c) \to Q,\ \neg\big((Q \leftrightarrow R) \vee \big(P(b) \to P(c)\big)\big)\big\}$. Conversion to CNF yields for the first three formulas

$$\begin{array}{ll} \neg P(b) \vee Q & (1) \\ \neg P(b) \vee R & (2) \\ \neg P(c) \vee Q & (3) \end{array}$$

For the fourth formula, we first replace the equivalence subformula by a disjunction of conjunctions (because of negative polarity). We obtain $\neg\big(\big((Q \wedge R) \vee (\neg Q \wedge \neg R)\big) \vee \big(P(b) \to P(c)\big)\big)$. By eliminating implications and pushing the negations downward, we get $\big(\big((\neg Q \vee \neg R) \wedge (Q \vee R)\big) \wedge \big(P(b) \wedge \neg P(c)\big)\big)$, that is, the clauses

$$\begin{array}{ll} \neg Q \vee \neg R & (4) \\ Q \vee R & (5) \\ P(b) & (6) \\ \neg P(c) & (7) \end{array}$$

We now apply the resolution calculus to (1)–(7). From (6) and (1) we obtain $Q$ (8), from (6) and (2) we obtain $R$ (9), from (8) and (4) we obtain $\neg R$ (10), and from (9) and (10) we obtain $\bot$. Since resolution is sound, the clause set is unsatisfiable, so the entailment holds.

*Grading scheme:* 4 points for the overall approach; 4 points for the CNF; 4 points for resolution.

## Assignment 6

**Part (a)** The clauses (2) and (3) are ground, so the only ground instance of (2) is (2) itself, and the only ground instance of (3) is (3) itself. The ground instances of (1) are those clauses that we obtain from $\neg P(x) \vee P(f(x))$ by replacing $x$ by a ground term. The set of ground terms is $\{b, f(b), f^2(b), \dots\}$, so the ground instances of (1) are $\{\neg P(f^i(b)) \vee P(f^{i+1}(b)) \mid i \in \mathbb{N}\}$. Comparing the largest literals in these clauses, it is obvious that the ground instances of (1) are ordered as follows:

$$\begin{array}{lll} & \neg P(b) \vee P(f(b)) & (1.1) \\ \prec & \neg P(f(b)) \vee P(f^2(b)) & (1.2) \\ \prec & \neg P(f^2(b)) \vee P(f^3(b)) & (1.3) \\ \prec & \neg P(f^3(b)) \vee P(f^4(b)) & (1.4) \\ \prec & \dots & \end{array}$$

We still have to figure out where to put clauses (2) and (3) in the clause ordering: $Q(b) \vee Q(f(b))$ is smaller than $\neg P(b) \vee P(f(b))$, and $\neg Q(f(b)) \vee P(f^3(b))$ comes between $\neg P(f(b)) \vee P(f^2(b))$ and $\neg P(f^2(b)) \vee P(f^3(b))$. So the ordering is

$$\begin{array}{lll} & Q(b) \vee Q(f(b)) & (3) \\ \prec & \neg P(b) \vee P(f(b)) & (1.1) \\ \prec & \neg P(f(b)) \vee P(f^2(b)) & (1.2) \\ \prec & \neg Q(f(b)) \vee P(f^3(b)) & (2) \\ \prec & \neg P(f^2(b)) \vee P(f^3(b)) & (1.3) \\ \prec & \neg P(f^3(b)) \vee P(f^4(b)) & (1.4) \\ \prec & \neg P(f^4(b)) \vee P(f^5(b)) & (1.5) \\ \prec & \neg P(f^5(b)) \vee P(f^6(b)) & (1.6) \\ \prec & \dots & \end{array}$$

*Grading scheme:* 3 points for the set of ground instances; 4 points for the ordering.

**Part (b)** Clause (3) produces $Q(f(b))$, clauses (1.1) and (1.2) are true in their own interpretations and produce nothing. Clause (2) produces $P(f^3(b))$, clause (1.3) is true in its own interpretation and produces nothing. All further clauses are productive and produce $P(f^4(b))$, $P(f^5(b))$, $P(f^6(b))$, … Since the construction does not fail, the limit $I^{\succ}_{G_\Sigma(N)} = \{Q(f(b))\} \cup \{P(f^i(b)) \mid i \geq 3\}$ is a model of $G_\Sigma(N)$.

*Grading scheme:* no errors: 7 points; one error: 4 points; two or more errors: 0 points.