Automated Reasoning I, 2015 Re-Exam, Sample Solution

Assignment 1

Part (a) Let $A = \{a, b\}$ with $\to = \{(a,a),(a,b)\}$. Every element of A has exactly one normal form, namely b. We get $L(a) = 1$ and $L(b) = 0$, so $a \to a$ but not $a \Rightarrow a$.

Part (b) We use induction over $L(b)$. If $L(b) = 0$, then the normal form of b with respect to $\to$ is b itself. Obviously, $b \Rightarrow^0 b$ and therefore $b \Rightarrow^* b$.

If $L(b) = n{+}1$, then there is a derivation with $n{+}1$ steps $b \to b'' \to^n b'$, where b' is the normal form of both b and b''. Clearly, there cannot exist any shorter derivation $b'' \to^m b'$ with $m < n$, since otherwise there would be a derivation $b \to^{m+1} b'$, contradicting the minimality assumption. So $L(b'') = n$, and therefore $b \Rightarrow b''$. By induction, $b'' \Rightarrow^* b'$, so $b \Rightarrow^* b'$.

Note:

- It is important to use a derivation $b \to^* b'$ *with minimal length*. For arbitrary derivations, the induction fails.

Part (c) Since $\to \supseteq \Rightarrow$, we get $\leftrightarrow^* \supseteq \Leftrightarrow^*$. To prove the reverse inclusion, we first show that $\to \subseteq \Leftrightarrow^*$. Assume that $a \to b$. Let c be the normal form of b. Clearly, c is also the normal form of a. By part (b), $a \Rightarrow^* c \Leftarrow^* b$, so $a \Leftrightarrow^* b$ as required. Since $\Leftrightarrow^*$ is reflexive, symmetric, and transitive, $\to \subseteq \Leftrightarrow^*$ implies $\leftrightarrow^* \subseteq \Leftrightarrow^*$.

Note:

- This property can also be proved by induction on the number of peaks (as in the proof of Newman's Lemma).

Assignment 2

(1) **true:** E.g., $U_\mathcal{A} = \{7, 8, 9\}$, $b_\mathcal{A} = 7$, $f_\mathcal{A}(7) = 8$, $f_\mathcal{A}(8) = 8$, $f_\mathcal{A}(9) = 9$, $P_\mathcal{A} = \{8\}$.
(2) **false:** If $f_\mathcal{A}(a) = a$ for every $a \in U_\mathcal{A}$, then $f_\mathcal{A}(f_\mathcal{A}(b_\mathcal{A})) = f_\mathcal{A}(b_\mathcal{A}) = b_\mathcal{A}$, but $f_\mathcal{A}(f_\mathcal{A}(b_\mathcal{A})) \in P_\mathcal{A}$ and $b_\mathcal{A} \notin P_\mathcal{A}$.
(3) **false:** F has infinitely many models.
(4) **true:** In every model of F, $P(x)$ holds for the assignment that maps x to $f_\mathcal{A}(f_\mathcal{A}(b_\mathcal{A}))$.
(5) **false:** E.g., in the model given for (1), $P(f(f(x)))$ does not hold for the assignment that maps x to 9.
(6) **true:** The universe of a Herbrand interpretation over Σ is the set of ground Σ-terms, i.e., $\mathrm{T}_\Sigma(\emptyset) = \{b, f(b), f(f(b)), f(f(f(b))), \dots\}$. Since the universe is infinite, there are infinitely many ways to interpret P.
(7) **false:** For every Herbrand model of F over Σ, the universe is infinite, see (6).
(8) **true:** In fact, *every* Herbrand model over Σ has an infinite universe, see (6).
(9) **true:** In every Herbrand model for F, $P(b)$ must be false and $P(f^n(b))$ must be true for every $n \geq 2$. Since $P(f(b))$ can be either true or false, there are two Herbrand models for F.

Assignment 3

By Lemma 3.21, it is sufficient to show that $\mathcal{A} \models (s \approx t)$ implies $\mathcal{B} \models (s \approx t)$. Assume that $\mathcal{A} \models (s \approx t)$. Let $\beta : X \to U_\mathcal{B}$ be an arbitrary $\mathcal{B}$-assignment. We have to show that $\mathcal{B}, \beta \models (s \approx t)$, that is, $\mathcal{B}(\beta)(s) = \mathcal{B}(\beta)(t)$. Since ϕ is surjective, there exists some $\mathcal{A}$-assignment $\alpha : X \to U_\mathcal{A}$ such that $\phi(\alpha(y)) = \beta(y)$ for every variable $y \in X$.

By induction over the structure of terms, we first show that $\phi(\mathcal{A}(\alpha)(u)) = \mathcal{B}(\beta)(u)$ for every term u: If $u = x$, then

$$\phi(\mathcal{A}(\alpha)(x)) = \phi(\alpha(x)) = \beta(x) = \mathcal{B}(\beta(x)).$$

Otherwise $u = f(u_1, \dots, u_n)$, then

$$\begin{aligned}
&\phi(\mathcal{A}(\alpha)(f(u_1, \dots, u_n))) \\
&\quad = \phi(f_\mathcal{A}(\mathcal{A}(\alpha)(u_1), \dots, \mathcal{A}(\alpha)(u_n))) \\
&\quad = f_\mathcal{B}(\phi(\mathcal{A}(\alpha)(u_1)), \dots, \phi(\mathcal{A}(\alpha)(u_n))) \\
&\quad = f_\mathcal{B}(\mathcal{B}(\beta)(u_1), \dots, \mathcal{B}(\beta)(u_n)) \\
&\quad = \mathcal{B}(\beta)(f(u_1, \dots, u_n)).
\end{aligned}$$

The main statement follows now from the fact that $\mathcal{A}, \alpha \models (s \approx t)$ for every assignment α, hence $\mathcal{A}(\alpha)(s) = \mathcal{A}(\alpha)(t)$, hence $\mathcal{B}(\beta)(s) = \phi(\mathcal{A}(\alpha)(s)) = \phi(\mathcal{A}(\alpha)(t)) = \mathcal{B}(\beta)(t)$.

Assignment 4

$C_1 = \neg P(c) \lor P(f(c))$.
C_1 is a ground instance of (3) and it is entailed by (2), which is smaller than C_1. In fact, C_1 is the only ground instance of a clause in N that is redundant w.r.t. N.

$C_2 = P(b)$.
C_2 is the smallest ground instance of a clause in N, namely (1). It is not entailed by smaller ground instances and therefore not redundant.
$C_3 = \neg P(c) \vee P(c)$.
C_3 is a tautology and the smallest redundant clause w.r.t. N.
$C_4 = P(c)$.
C_4 is the smallest non-empty Σ-clause. It is neither a ground instance of a clause in N nor entailed by smaller ground instances (and therefore not redundant).

Assignment 5

Part (a) We construct a strict tableau for the *negated* input formula (1):

$$\neg\big((P \to Q) \to ((P \vee R) \to (Q \vee R))\big) \quad (1)$$
$$P \to Q \quad (2)$$
$$\neg((P \vee R) \to (Q \vee R)) \quad (3)$$
$$P \vee R \quad (4)$$
$$\neg(Q \vee R) \quad (5)$$
$$\neg Q \quad (6)$$
$$\neg R \quad (7)$$

$\neg P$ (8) $\qquad$ Q (9)

P (10) $\qquad$ R (11)

The α-expansion of (1) yields (2) and (3), α-expansion of (3) yields (4) and (5), α-expansion of (5) yields (6) and (7), β-expansion of (2) yields (8) and (9), and β-expansion of (4) yields (10) and (11). Since every path is now closed, the negated input formula is unsatisfiable, so the input formula is valid.

Part (b) We construct a strict tableau for the *negated* input formula:

$$\neg\big((P \vee Q) \to (P \wedge Q)\big) \quad (1)$$
$$P \vee Q \quad (2)$$
$$\neg(P \wedge Q) \quad (3)$$

P (4) $\qquad$ Q (5)

$\neg P$ (6) $\qquad$ $\neg Q$ (7) $\qquad$ $\neg P$ (8) $\qquad$ $\neg Q$ (9)

The α-expansion of (1) yields (2) and (3), β-expansion of (2) yields (4) and (5), β-expansion of (3) yields (6) and (7), and once more β-expansion of (3) yields (8) and (9). Since the second (and also the third) path is now *maximal and open,* the set of formulas on this path is satisfiable. In particular, the negated input formula is satisfiable, so the input formula is not valid.

Note:

- It is not possible to test validity by checking whether all paths in a maximal tableau for the *non-negated* formula are open. E.g., $F = (P \vee \neg P) \vee (Q \wedge \neg Q)$ is valid, but the maximal strict tableau for F has a closed path, whereas $G = (P \vee Q)$ is not valid, but the maximal strict tableau for G has only open paths.

Assignment 6

We start with the three given equations (1)–(3)

$f(g(x), x) \approx b$	(1)	$f(g(x), x) \to b$	(4)
$f(x, b) \approx x$	(2)	$f(x, b) \to x$	(5)
$g(h(x)) \approx x$	(3)	$g(h(x)) \to x$	(6)
$b \approx g(b)$	(7)	$g(b) \to b$	(8)
$b \approx f(x, h(x))$	(9)	$f(x, h(x)) \to b$	(10)
$b \approx f(b, b)$	(11)		
$b \approx b$	(12)		

By applying "Orient" three times, we replace (1)–(3) by the corresponding rewrite rules (4)–(6). Using the critical pair between rules (4) and (5), the "Deduce" rule adds equation (7). The "Orient" rule replaces equation (7) by rule (8). Using the critical pair between rules (4) and (6), the "Deduce" rule adds equation (9). The "Orient" rule replaces equation (9) by rule (10). Using the critical pair between rules (4) and (8), the "Deduce" rule adds equation (11). The "Simplify-Eq" rule uses the rewrite rule (5) to replace equation (11) by equation (12). Equation (12) is trivial, so it can be eliminated using "Delete". Since all critical pairs between persisting rules have been computed and all equations have been eliminated, we can stop now; the final rewrite system is $\{(4), (5), (6), (8), (10)\}$.