Assignment 1 *(Propositional Logic)* (12 points)

Let N be a set of propositional clauses. Prove or refute the following statement: If N contains clauses $C_i \vee D_i$ $(1 \leq i \leq n)$ such that $\{ C_i \mid 1 \leq i \leq n \} \models \bot$, then $N \models \bigvee_{1 \leq i \leq n} D_i$.

Assignment 2 *(Algebras, Herbrand Interpretations)* (12 points)

Decide for each of the following statements whether it is true or false:

(1) If $\Sigma = (\{b/0,\ c/0\},\ \{P/1\})$, then $P(b) \vee \neg P(c)$ has exactly three Herbrand models over Σ.

(2) If $\Sigma = (\{f/1,\ c/0\},\ \{P/1\})$, then $P(c) \vee P(f(c))$ has an Herbrand model over Σ whose universe has exactly four elements.

(3) If $\Sigma = (\{f/1,\ c/0\},\ \{P/1\})$, then $\neg P(c) \wedge \forall x\, P(f(x))$ has a model whose universe has exactly five elements

(4) If $\Sigma = (\{b/0,\ c/0,\ d/0\},\ \{P/1\})$, then $P(b) \vee \neg P(b)$ and $P(c) \vee \neg P(d)$ are equisatisfiable.

(5) If $\Sigma = (\{f/1,\ c/0\},\ \{P/1\})$, N is a set of universally quantified Σ-clauses, and every clause in N has at least one positive literal, then N has an Herbrand model.

(6) If $\Sigma = (\{f/1,\ c/0\},\ \{P/1\})$, N is a set of universally quantified Σ-clauses, and $N \models \neg P(x) \vee P(f(x))$, then N has a model.

(7) If $\Sigma = (\{f/1,\ c/0\},\ \{P/1\})$, then $\forall x\, P(f(x)) \models \forall y\, P(c) \vee P(f(f(y)))$.

(Note on grading: A yes/no answer is sufficient; you do not have to give an explanation. However, you need at least four correct answers to get any points for this assignment. Missing answers count like false answers.)

Assignment 3 *(Lexicographic Path Ordering)* (10 points)

Let $\Sigma = (\Omega, \emptyset)$ with $\Omega = \{f/1,\ g/1,\ h/2,\ b/0,\ c/0\}$. Find a total precedence $\succ$ on Ω such that the lexicographic path ordering $\succ_{\text{lpo}}$ that is induced by $\succ$ (with left-to-right status for h) satisfies the following three properties simultaneously.

$$h(x, f(y)) \succ_{\text{lpo}} h(g(y), y) \qquad (1)$$
$$h(x, c) \succ_{\text{lpo}} g(h(x, b)) \qquad (2)$$
$$g(b) \succ_{\text{lpo}} c \qquad (3)$$

Assignment 4 *(Constructing an Interpretation)* (7 + 7 = 14 points)

Let $\Sigma = (\Omega, \Pi)$ be a signature with $\Omega = \{f/1,\ b/0,\ c/0\}$ and $\Pi = \{P/1\}$. Suppose that the atom ordering $\succ$ is a Knuth-Bendix ordering with weight 1 for all predicate symbols, function symbols, and variables, and with the precedence $P > f > b > c$. Let N be the set $\{C_1, C_2, C_3\}$.

$$\begin{aligned} C_1 &= P(b) \\ C_2 &= \neg P(f(f(c)) \\ C_3 &= P(x) \vee P(f(x)) \end{aligned}$$

Part (a) Sketch how the set $G_\Sigma(N)$ of all ground instances of clauses in N looks like. How is it ordered with respect to the clause ordering $\succ_C$?

Part (b) Construct the candidate interpretation $I^{\succ}_{G_\Sigma(N)}$ of the set of all ground instances of clauses in N. Which clauses in $G_\Sigma(N)$ are productive and what do they produce?

Assignment 5 *(Termination, Confluence)* (10 + 6 = 16 points)

Let $\Sigma = (\Omega, \emptyset)$ be a finite signature; let $f/1 \in \Omega$; let $\succ$ be a simplification ordering; let R be a confluent term rewrite system contained in $\succ$.

Part (a) Prove: If f does not occur in any left-hand side of a rule in R, then $R \cup \{f(f(x)) \to f(x)\}$ is terminating and confluent.

Part (b) Give an example that shows that $R \cup \{f(f(x)) \to f(x)\}$ need not be confluent if f occurs in a left-hand side of a rule in R.

Assignment 6 *(Knuth-Bendix Completion)* (16 points)

Let E be the following set of equations over $\Sigma = (\{f/1,\ g/1,\ h/1\}, \emptyset)$.

$$f(g(f(x))) \approx h(x) \qquad (1)$$
$$g(h(x)) \approx x \qquad (2)$$

Apply the Knuth-Bendix completion procedure to E and transform it into a finite convergent term rewrite system; use the Knuth-Bendix ordering with weight 1 for all function symbols and variables and the precedence $f > g > h$. Use a reasonable strategy.