Assignment 1 *(Multisets)* (12 + 8 = 20 points)

Let M be a set, let $\succ$ be a strict partial ordering over M. Let $b, b_1, b_2 \in M$ and let S, S_1, S_2 be finite multisets over M.

Part (a)

Prove or refute: If $\{b\} \succ_{\text{mul}} S_1$ and $\{b\} \succ_{\text{mul}} S_2$, then $\{b\} \succ_{\text{mul}} S_1 \cup S_2$.

Part (b)

Prove or refute: If $S \succ_{\text{mul}} \{b_1\}$ and $S \succ_{\text{mul}} \{b_2\}$, then $S \succ_{\text{mul}} \{b_1, b_2\}$.

Assignment 2 *(First-order logic, Semantics)* (10 points)

Let $\Sigma = (\Omega, \Pi)$ be a first-order signature with $\Omega = \{f/1,\ b/0,\ c/0\}$ and $\Pi = \{P/1\}$. Are the following statements correct?

(1) The formula $\forall x\, P(x)$ has infinitely many Σ-models.

(2) Every model of $\forall x\, P(x)$ is a model of $\forall x\, P(f(x))$.

(3) The formula $\neg P(b) \wedge \forall x\, P(x)$ has a Σ-model with an infinite universe.

(4) The formula $\neg P(b) \wedge \forall x\, P(f(x))$ has a Σ-model with a two-element universe.

(5) Every Σ-model of $P(b) \wedge P(c) \wedge \forall x\, P(f(x))$ is a model of $\forall x\, P(x)$.

(6) Every Herbrand model over Σ of $P(b) \wedge P(c) \wedge \forall x\, P(f(x))$ has an infinite universe.

(7) The formula $P(b) \vee P(c)$ has exactly three Herbrand models over Σ.

(8) The formula $\forall x\, P(f(x))$ has exactly four Herbrand models over Σ.

(Note on grading: A yes/no answer is sufficient; you do not have to give an explanation. However, you need at least five correct answers in this assignment to get any points. Missing answers count like false answers.)

Assignment 3 *(Tableaux)* (10 points)

Check the satisfiability of the following set of formulas using the tableau calculus:

$$P(b) \wedge \neg P(d) \qquad (1)$$

$$P(c) \vee \big(P(b) \wedge P(d)\big) \qquad (2)$$

$$P(c) \rightarrow \neg\big(P(b) \vee P(d)\big) \qquad (3)$$

Use exactly the expansion rules given in the lecture. Do not use shortcuts, do not preprocess the formulas by converting them to CNF, DNF, or NNF. State explicitly whether the set is satisfiable or not and give an explanation for that statement.

Assignment 4 *(Equality)* (8 points)

Let $\Sigma = (\Omega, \emptyset)$ with $\Omega = \{b/0,\ f/1,\ g/1\}$; let E be the set of (implicitly universally quantified) equations $\{\, f(g(f(x))) \approx b \,\}$.
Give one possible derivation for the statement $E \vdash f(g(b)) \approx b$.

Assignment 5 *(Knuth-Bendix completion)* (14 points)

Let E be the following set of equations over $\Sigma = (\{f/2,\ g/1,\ b/0,\ c/0\}, \emptyset)$.

$$f(x,x) \approx f(x,b) \qquad (1)$$
$$f(x,x) \approx f(c,x) \qquad (2)$$
$$f(x,x) \approx g(x) \qquad (3)$$

Apply the Knuth-Bendix completion procedure to E and transform it into a finite convergent term rewrite system; use a Knuth-Bendix ordering with weight 1 for all function symbols and variables and the precedence $f > g > b > c$. Use a reasonable strategy.

Assignment 6 *(Orderings, Dependency pairs)* (9 + 9 = 18 points)

Part (a)

Give an example of a term rewrite system R with exactly one rewrite rule such that

(1) $\rightarrow_R$ is not contained in any LPO $\succ$, and

(2) the approximated dependency graph for R contains exactly one node and no edges.

Give a brief explanation.

Part (b)

Give an example of a term rewrite system R with exactly one rewrite rule such that

(1) $\rightarrow_R$ is contained in some LPO $\succ$, and

(2) the approximated dependency graph for R contains exactly one node and one edge.

Give a brief explanation.