Assignment 1 *(First-Order Logic)* (12 points)

Let $\Sigma = (\Omega, \Pi)$ be a signature. Let $P/1$ and $Q/0$ be predicate symbols in Π. Let N be a set of (universally quantified) clauses over Σ. Let N_0 be the set of all clauses in N that contain a literal $\neg P(t)$ for some $t \in \mathrm{T}_\Sigma(X)$; let N_1 be the set of all clauses in N that contain a literal $P(t')$ for some $t' \in \mathrm{T}_\Sigma(X)$. Prove: If all clauses in $N_0 \setminus N_1$ contain also the literal $\neg Q$ and if all clauses in $N_1 \setminus N_0$ contain also the literal Q, then N and $N \setminus N_0 \setminus N_1$ are equisatisfiable.

Assignment 2 *(Clause Normal Forms)* (9 points)

Section 3.6 of the lecture notes describes rules to transform a first-order formula into CNF. These include the following set of "Miniscoping" rules:

> Apply the reduction system $\Rightarrow_{\mathrm{MS}}$ modulo associativity and commutativity of $\wedge$, $\vee$. For the rules below we assume that x occurs freely in F, F', but x does not occur freely in G:
>
> $$\begin{array}{lll} H[\mathsf{Q}x\,(F \wedge G)]_p & \Rightarrow_{\mathrm{MS}} & H[(\mathsf{Q}x\,F) \wedge G]_p \\ H[\mathsf{Q}x\,(F \vee G)]_p & \Rightarrow_{\mathrm{MS}} & H[(\mathsf{Q}x\,F) \vee G]_p \\ H[\forall x\,(F \wedge F')]_p & \Rightarrow_{\mathrm{MS}} & H[(\forall x\,F) \wedge (\forall x\,F')]_p \\ H[\exists x\,(F \vee F')]_p & \Rightarrow_{\mathrm{MS}} & H[(\exists x\,F) \vee (\exists x\,F')]_p \\ H[\mathsf{Q}x\,G]_p & \Rightarrow_{\mathrm{MS}} & H[G]_p \end{array}$$

None of these rules is applicable to the formula

$$P(b) \vee \big(\forall w\ \exists x\ \exists y\ \exists z\ (\ R(x,y,z) \wedge R(y,z,w)\ )\big)$$

However, one can extend the rules above by more powerful rules that are applicable to this formula. Give an example of such a rule. Note that all rules must be satisfiability-preserving. (Hint: Think about $\exists x$.)

Assignment 3 *(Ordered Resolution)* (12 + 4 = 16 points)

Let $\Sigma = (\{f/1,\ g/1,\ h/1,\ b/0,\ c/0\},\ \{P/2,\ Q/1,\ R/2\})$; let N be the following set of clauses over Σ:

$$P(f(x),x)\ \vee\ P(c,x)\ \vee\ R(g(x),x) \qquad (1)$$

$$\neg P(y,f(y)) \qquad (2)$$

$$\neg P(y,c)\ \vee\ \neg P(z,h(y))\ \vee\ Q(z) \qquad (3)$$

$$Q(b)\ \vee\ Q(x)\ \vee\ \neg R(g(x),x) \qquad (4)$$

$$R(g(c),y) \qquad (5)$$

Part (a) Suppose that the atom ordering $\succ$ is a lexicographic path ordering with the precedence $P > Q > R > f > g > h > b > c$ and that the selection function *sel* selects no literals. Compute all $\mathrm{Res}^{\succ}_{sel}$ inferences between the clauses (1)–(5). (Do not compute inferences between derived clauses. Do not compute any inferences that violate the restrictions of the calculus.)

Part (b) One of the conclusions of the inferences computed in part (a) is (relatively obviously) redundant w.r.t. N. Which one? Why?

Assignment 4 *(Rewriting, E-Algebras)* (3 + 5 + 6 = 14 points)

Let $\Sigma = (\Omega, \emptyset)$, let $\Omega = \{f/1,\ b/0,\ c/0,\ d/0\}$. Let E be the set of equations $\{f(b) \approx d,\ f(c) \approx d,\ f(f(x)) \approx f(x)\}$. Let X be a countably infinite set of variables.

Part (a) Show that $f(d) \leftrightarrow_E^* d$.

Part (b) Sketch how the universe of $\mathrm{T}_\Sigma(\emptyset)/E$ looks like. How many elements does it have?

Part (c) Decide for each of the following equations whether it holds in $\mathrm{T}_\Sigma(X)/E$ and whether it holds in $\mathrm{T}_\Sigma(\emptyset)/E$. Give a very brief explanation.

$$f(b) \approx b \qquad (1)$$

$$\forall y \;\; f(f(f(y))) \approx f(f(y)) \qquad (2)$$

$$\forall x\, \forall y \;\; f(x) \approx f(y) \qquad (3)$$

Assignment 5 *(Confluence)* (8 + 8 = 16 points)

Let $\Sigma = (\Omega, \emptyset)$ be a signature; let R be a term rewrite system.

Part (a) Prove: If $s \to_R t$, then $\mathrm{var}(s) \supseteq \mathrm{var}(t)$.

Part (b) Prove: If $x \in X$ is a variable, $s \in \mathrm{T}_\Sigma(X)$ is a term such that $x \notin \mathrm{var}(s)$, and $R \models x \approx s$, then R is not confluent.

Assignment 6 *(Dependency Pairs)* (4 + 6 + 3 = 13 points)

Let R be the set of rewrite rules

$$f(p(x)) \to p(h(q(x))) \qquad (1)$$

$$f(q(x)) \to p(p(x)) \qquad (2)$$

$$f(f(x)) \to f(x) \qquad (3)$$

$$g(p(x)) \to h(f(x)) \qquad (4)$$

$$g(q(g(x))) \to g(b) \qquad (5)$$

$$h(p(x)) \to g(c) \qquad (6)$$

$$h(q(q(x))) \to g(q(x)) \qquad (7)$$

Part (a) Determine $\mathrm{DP}(R)$. (Use a numbering scheme for the dependency pairs that indicates from which rewrite rules they are derived, that is, denote dependency pairs derived from, say, rule (4) by (4a), (4b), ...).

Part (b) Construct the overapproximated dependency graph for R (using cap and ren). Use it to show that R is terminating.

Part (c) There is one edge in the overapproximated dependency graph for R that is not present in the (exact) dependency graph for R. Which one? Explain.