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# Tutorials for "Automated Reasoning"
## Exercise sheet 1

**Exercise 1.1:**
(a) Find an abstract reduction system $(A, \rightarrow)$, such that the relations $\rightarrow$, $\leftrightarrow$, $\leftrightarrow^+$, and $\leftrightarrow^*$ are all different.

(b) Find an abstract reduction system $(B, \rightarrow)$, such that $\rightarrow^+$ is irreflexive and $\rightarrow$ is normalizing, but not terminating.

**Exercise 1.2:**
For an alphabet $\Sigma$ with a well-founded ordering $>_\Sigma$ let the relation $>_{\Sigma,\text{lex}} \subseteq \Sigma^* \times \Sigma^*$ be defined by $w >_{\Sigma,\text{lex}} w'$ if and only if $w$ and $w'$ have the same length $n$ and $w$ is larger than $w'$ in the $n$-fold lexicographic combination of $>_\Sigma$. Let the relation $\succ$ be defined as

$$w \succ w' :\Leftrightarrow |w| > |w'| \text{ or } (|w| = |w'| \text{ and } w >_{\Sigma,\text{lex}} w').$$

Prove that $\succ$ is a well-founded ordering on $\Sigma^*$. (Note: We define the 0-fold lexicographic combination of an ordering as $\emptyset$ and the 1-fold lexicographic combination of an ordering as the ordering itself. You may use the fact that for any $n \in \mathbb{N}$ the $n$-fold lexicographic combination of a well-founded ordering is well-founded.)

**Exercise 1.3:**
Let $(M, \succ)$ be an ordering and $b, c \in M$. We say that $b$ is a *successor* of $c$, if $b \succ c$ and if there exists no $d \in M$ with $b \succ d$ and $d \succ c$.

(1) Prove: If $\succ$ is well-founded, then every element of $M$ has either a successor or it is maximal.

(2) Prove: If $\succ$ is well-founded and total, then every element of $M$ has at most one successor.

(3) Give an example of a set $M$, a well-founded ordering $\succ$ on $M$, and an element $b \in M$ such that $b$ is neither a minimal element of $M$ nor a successor of any other element of $M$.

**Exercise 1.4:**
You are asked to review a scientific article that has been submitted to a conference on automated reasoning. On page 3 of the article, the authors write the following:

> **Theorem 2.** Let $\rightarrow_1$ and $\rightarrow_2$ be two binary relations over a non-empty set $M$. If $\rightarrow_1$ and $\rightarrow_2$ are terminating, then $\rightarrow_1 \cup \rightarrow_2$ is also terminating.
>
> **Proof.** Since $\rightarrow_1$ is terminating, $\rightarrow_1^+$ is a well-founded ordering. Assume that there exists an infinite descending $(\rightarrow_1 \cup \rightarrow_2)$-chain. Since $\rightarrow_1^+$ is well-founded, there exists a minimal element $b$ with respect to $\rightarrow_1^+$ such that there is an infinite descending $(\rightarrow_1 \cup \rightarrow_2)$-chain starting with $b$.
>
> Case 1: The $(\rightarrow_1 \cup \rightarrow_2)$-chain starts with a $\rightarrow_1$-step $b \rightarrow_1 b'$. The rest of the chain, starting with $b'$, is still infinite. However, $b'$ is smaller than $b$ with respect to $\rightarrow_1^+$. This contradicts the minimality of $b$.
>
> Case 2: The $(\rightarrow_1 \cup \rightarrow_2)$-chain starts with a $\rightarrow_2$-step $b \rightarrow_2 b'$. Since $\rightarrow_2$ is terminating, the chain cannot consist only of $\rightarrow_2$-steps. Therefore there must be some $\rightarrow_1$-step in the chain, say $b'' \rightarrow_1 b'''$. Hence there exists an infinite $(\rightarrow_1 \cup \rightarrow_2)$-chain starting with this step. But as we have seen in Case 1, an infinite $(\rightarrow_1 \cup \rightarrow_2)$-chain cannot start with a $\rightarrow_1$-step. So there is again a contradiction.
>
> Consequently, every descending $(\rightarrow_1 \cup \rightarrow_2)$-chain must be finite, which means that $\rightarrow_1 \cup \rightarrow_2$ is terminating.

(1) Is the "proof" correct (yes/no)?

(2) If the "proof" is not correct:

  (a) Which step is incorrect?

  (b) Does the "theorem" hold? If yes, give a correct proof, otherwise give a counterexample.

Bring your solution to the tutorial on November 8 and compare it with the solution that is discussed there. If you are still unsure afterwards whether your solution is correct or not, feel free to ask the instructor after the tutorial. Your solution will not be graded.