Automated Reasoning I, 2021/22
Re-Exam, Sample Solution

Assignment 1

Part (a) Assume that $>$ is well-founded and that b is the only element of M that is minimal in M, but that b is not the smallest element of M. Let $X = \{\, x \in M \mid b \leq x \,\}$ and let $Y = M \setminus X$. Since b is not the smallest element of M, we know that $Y \neq \emptyset$. Since $>$ is well-founded, this implies that there exists some $c \in Y$ that is minimal in Y. By assumption, b is the only element of M that is minimal in M, so c is not minimal in M. Therefore, there exists some $d \in M$ such that $d < c$. Since c is minimal in Y, d cannot be contained in Y. But then $d \in X$, which implies $b \leq d < c$ and thus $c \in X$, contradicting the fact that $c \in Y$.

Part (b) Let $M = \{\, x \in \mathbb{Z} \mid x \leq 0 \,\} \cup \{b\}$, where $>$ is the usual ordering on integer numbers and b is incomparable with all integer numbers. Then b is minimal in M (since no element of M is smaller), and it is the only minimal element of M (since for every other $x \in M$ there exists a smaller element $x - 1 \in M$, but b is not the smallest element of M, since the other elements of M are not larger than b.

Assignment 2

(1): **true:** $P_{\mathcal{A}}$ cannot equal $U_{\mathcal{A}}$, since $b_{\mathcal{A}} \notin P_{\mathcal{A}}$; $P_{\mathcal{A}}$ cannot be empty, since $f_{\mathcal{A}}(f_{\mathcal{A}}(b_{\mathcal{A}})) \in P_{\mathcal{A}}$.
(2) **true:** Let $U_{\mathcal{A}} = \{7, 8, 9\}$, let $b_{\mathcal{A}} = 7$, let $f_{\mathcal{A}}$ map every element of $U_{\mathcal{A}}$ to 8, and let $P_{\mathcal{A}} = \{8\}$.
(3) **true:** See (2).
(4) **false:** F has infinitely many Σ-models; in particular it has Σ-models with any universe with at least 2 elements.
(5) **true:** Since $\mathrm{T}_\Sigma(\emptyset)$ is infinite, there are infinitely many different possibilities to choose a subset $P_{\mathcal{A}} \subseteq \mathrm{T}_\Sigma(\emptyset)$.
(6) **false:** All Herbrand models of F over Σ have the same universe $\mathrm{T}_\Sigma(\emptyset)$ (which is infinite).
(7) **false:** If $\mathcal{A}$ is an Herbrand model over Σ, then $\mathcal{A}(\beta)(t) = t$ for every ground term $t \in \mathrm{T}_\Sigma(\emptyset)$, so $\mathcal{A}(\beta)(f(b))$ and $\mathcal{A}(\beta)(f(f(b)))$ are different elements of the universe.

Grading scheme: 4th, 5th, 6th, 7th correct answer: 3 points each.

Assignment 3

There are three critical pairs:

between (1) at position 1 and a renamed copy of (1):
$\sigma = \{x \mapsto f(x')\}$,
$h(h(f(x'))) \leftarrow f(f(f(x'))) \rightarrow f(h(h(x')))$,
critical pair: $\langle h(h(f(x'))), f(h(h(x')))\rangle$.

between (2) at position 1 and a renamed copy of (1):
$\sigma = \{y \mapsto f(x')\}$,
$g(f(x'), x) \leftarrow g(f(f(x')), x) \rightarrow g(h(h(x')), x)$,
critical pair: $\langle g(f(x'), x), g(h(h(x')), x)\rangle$.

between (3) at position 1 and (2):
$\sigma = \{z \mapsto f(y),\ x \mapsto f(c)\}$,
$f(f(y)) \leftarrow h(g(f(y), f(c))) \rightarrow h(g(y, f(c)))$,
critical pair: $\langle f(f(y)), h(g(y, f(c)))\rangle$.

Note: The rules (1) and (2) are not variable-disjoint. To compute the critical pair between (2) at position 1 and (1), it is necessary to rename the variable x in either (1) or (2), even though the term $f(f(x))$ and the subterm $f(y)$ of $g(f(y), x)$ are unifiable already without the renaming.

Grading scheme: -4 points for each missing or incorrect critical pair; -2 points for small mistakes.

Assignment 4

First we observe that $h(x, \ldots, x)$ is larger than its proper subterm x in every simplification ordering $\succ$. Therefore $l \succ r$ holds in fact for all $l \rightarrow r \in R \cup \{\, h(x, \ldots, x) \rightarrow x \,\}$. Consequently, $R \cup \{\, h(x, \ldots, x) \rightarrow x \,\}$ is terminating.

Second, we observe that the rewrite rule $h(x, \ldots, x) \rightarrow x$ has neither a critical pair with itself, nor with any rule $l \rightarrow r \in R$ (since h does not occur in l). Consequently, every critical pair between rules in $R \cup \{\, h(x, \ldots, x) \rightarrow x \,\}$ is a critical pair between rules in R. Since R is confluent, all critical pairs between rules in R are joinable in R, and hence also joinable in $R \cup \{\, h(x, \ldots, x) \rightarrow x \,\}$.

Using the critical pair theorem we conclude that $R \cup \{ h(x, \ldots, x) \to x \}$ is locally confluent; and since it is terminating, it is also confluent.

Assignment 5

Part (a) $\to_R$ is contained in an LPO with the precedence $f > h > g$.

Part (b) $\to_R$ is not contained in any KBO, since the first rewrite rule has more occurrences of x in the right-hand side than in the left-hand side.

Part (c) $\to_R$ is contained in a polynomial ordering where the symbols in Σ are interpreted by $P_f(X_1) = 3X_1$, $P_g(X_2) = X_1 + 1$, $P_b = 1$, $P_c = 4$.

Grading scheme: 5 points for each correct ordering or explanation.

Assignment 6

Part (a)
Term 3: $g(h(*), h(*))$.
Term 5: $g(h(b), *)$.
Term 12: $f(g(*, b))$.

Grading scheme: 1 point per correct solution.

Part (b)
$g(*, h(*))$: Term 7.
$f(g(c, b))$: not contained in the index.
$g(h(*), b)$: Term 2.

Grading scheme: 1 point per correct solution.

Part (c)
$f(g(h(c), f(b)))$ is reducible by the rules whose left-hand sides have the numbers 9, 4, and 11.

Grading scheme: -2 points per incorrect or missing solution.