Automated Reasoning I, 2017
Midterm Exam, Sample Solution

Assignment 1

Part (a) With a reasonable strategy and the given literal selection rule, the CDCL procedure yields

$$\underset{}{P^{\mathrm{d}}}\ \underset{}{Q^{\mathrm{d}}}\ \underset{}{R^{\mathrm{d}}}\ \underset{(8)}{\neg S}\ \underset{(7)}{\neg U}\ \underset{(9)}{T} \parallel N$$

At this point, clause (10) is a conflict clause. By resolving (10) with (9), we obtain $\neg P \vee \neg R \vee S \vee U$ (which is not a backjump clause), by resolving this clause with (7), we obtain $\neg P \vee \neg R \vee S$ (which is not a backjump clause either), by resolving this clause with (8), we obtain $\neg P \vee \neg R$ (11), which is a backjump clause. The best possible successor state for this backjump clause is $P^{\mathrm{d}}\ \neg R \parallel N$. After learning clause (11), we continue and obtain

$$P^{\mathrm{d}}\ \underset{(11)}{\neg R}\ \underset{(5)}{\neg S}\ \underset{(6)}{T}\ \underset{(7)}{\neg U}\ \underset{(4)}{\neg Q} \parallel N \cup \{(11)\}$$

Since all literals are defined and all clauses in N are true, this is a final state, so by Thm. 2.19, the literals on the trail are a model of N.

Grading scheme: *Decide* and *Unit Propagation* steps: 3 points; computing the backjump clause according to the 1UIP strategy: 2 points; choosing a legal successor state for backjumping: 1 point; choosing the optimal successor state: 2 more points. (Participants who avoided backjumping completely by ignoring the given literal selection rule for *Decide* did not get the 5 points for backjumping.)

Part (b) In the final state of part (a), all literals on the trail except the initial P^{d} are deduced literals. By Lemma 2.16, all literals after P^{d} are implied by N and P: After making P true, we are *forced* to make R false, S false, T true, U false, and Q false. As there cannot exist any other model of N in which P is true, the model that we derived in part (a) is the only one.

Part (c) To show that there is no model of N in which P is false, we add the clause $\neg P$ (12) to N. With a reasonable strategy we get

$$\underset{(12)}{\neg P}\ \underset{(1)}{S}\ \underset{(5)}{R} \parallel N \cup \{(12)\}$$

At this point, clause (8) is a conflict clause. Since there are no decision literals, we can derive *fail*, so the clause set is unsatisfiable.

Assignment 2

Part (a) By Thm. 3.13, ground resolution is sound. That means that for every algebra $\mathcal{A}$, whenever the premises of an inference hold in $\mathcal{A}$, then the conclusion holds in $\mathcal{A}$ as well. In particular, if the premises are tautological (i.e., hold in every algebra $\mathcal{A}$), then the conclusion holds in every algebra $\mathcal{A}$, so it is also tautological. Thus, if all clauses in a set M are tautologies, then all clauses in $\mathit{Res}(M)$ are tautologies. By induction over n we can now show that, if all clauses in N are tautologies, then all clauses in $\mathit{Res}^n(N)$ are tautologies. So, all clauses in $\mathit{Res}^*(N) = \bigcup_{n\geq 0} \mathit{Res}^n(N)$ are tautologies. The reverse direction follows immediately from the fact that $N \subseteq \mathit{Res}^*(N)$.

Grading scheme: "⇒": 3 points; "⇐": 5 points.

Part (b) The statement does not hold. For instance, if $N = \{\, P \vee Q,\ \neg P,\ \neg Q \,\}$ then $\mathit{Res}(N) = \{\, Q,\ P \}$; if $N = \{\, \neg P \vee \neg P,\ P \,\}$ then $\mathit{Res}(N) = \{\, \neg P\}$; and if $N = \{\, \bot \,\}$ then $\mathit{Res}(N) = \emptyset$. In all three examples, the set N is unsatisfiable, but $\mathit{Res}(N)$ is satisfiable.

Assignment 3

The "only if" part is trivial. For the "if" part suppose that N' is satisfiable, that is, there is a valuation $\mathcal{B}$ such that $\mathcal{B}(C) = 1$ for every $C \in N'$. Define a valuation $\mathcal{A}$ by $\mathcal{A}(P) = 1$ if $P \in S$, $\mathcal{A}(P) = 0$ if $\neg P \in S$, and $\mathcal{A}(P) = \mathcal{B}(P)$ otherwise. Since every clause in N contains some literal of S, $\mathcal{A}(C) = 1$ for every $C \in N$. For a clause $C \in N'$ we distinguish two cases: If C contains some literal of S, then again $\mathcal{A}(C) = 1$. Otherwise C contains neither a literal in S nor the complement of a literal in S, so $\mathcal{A}(C) = \mathcal{B}(C)$. Since $\mathcal{B}(C) = 1$ for every $C \in N'$, we get $\mathcal{A}(C) = 1$ for every $C \in N'$.

Grading scheme: "$\Rightarrow$": 3 points; "$\Leftarrow$": 9 points.

Assignment 4

(1) **true:** Define $b_\mathcal{A} = 7$, $f_\mathcal{A}(a) = 8$ for $a \in \{7, 8, 9\}$ and $P_\mathcal{A} = \{7\}$.
(2) **false:** If $f_\mathcal{A}(a) = a$ for every $a \in U_\mathcal{A}$, then $b_\mathcal{A} = f_\mathcal{A}(b_\mathcal{A}) = f_\mathcal{A}(f_\mathcal{A}(b_\mathcal{A}))$, but $b_\mathcal{A} \in P_\mathcal{A}$ and $f_\mathcal{A}(f_\mathcal{A}(b_\mathcal{A})) \notin P_\mathcal{A}$.
(3) **true:** Define $P_\mathcal{A} = \{b\}$.
(4) **false:** The formula is contradictory; it has no model and in particular no Herbrand model.
(5) **false:** Every Herbrand interpretation (and therefore every Herbrand model) over the signature Σ has the infinite universe $\mathrm{T}_\Sigma = \{b, f(b), f(f(b)), \dots\}$.
(6) **true:** The Herbrand interpretation in which $P_\mathcal{A} = \mathrm{T}_\Sigma$ is the only Herbrand model.
(7) **true:** If $f_\mathcal{A}(a) \in P_\mathcal{A}$ for every $a \in U_\mathcal{A}$, then $f_\mathcal{A}(f_\mathcal{A}(a)) \in P_\mathcal{A}$ for every $a \in U_\mathcal{A}$.

Grading scheme: 4th, 5th, 6th, 7th correct answer: 3 points each.

Assignment 5

The NNF transformation of

$$\exists v\, \forall x\, \forall y\, \forall z\, \neg \forall w\, \big(\neg P(c, w, z, x) \wedge Q(w, y, f(x), v)\big)$$

yields

$$\exists v\, \forall x\, \forall y\, \forall z\, \exists w\, \big(P(c, w, z, x) \vee \neg Q(w, y, f(x), v)\big)$$

Miniscoping proceeds bottom-up. We start with moving $\exists w$ inside the disjunction and get

$$\exists v\, \forall x\, \forall y\, \forall z\, \big(\exists w\, P(c, w, z, x) \vee \exists w\, \neg Q(w, y, f(x), v)\big)$$

We move $\forall z$ inside the disjunction (which is possible since z occurs only in the first subformula), then we move $\forall y$ inside the disjunction (which is possible since y occurs only in the second subformula). We obtain

$$\exists v\, \forall x\, \big(\forall z\, \exists w\, P(c, w, z, x) \vee \forall y\, \exists w\, \neg Q(w, y, f(x), v)\big)$$

As x occurs in both subformulas, the quantifier $\forall x$ may not be moved inside the disjunction. The quantifier $\exists v$ is located in front of a universally quantified formula, not in front of a disjunction, so it may not be moved either. Variable renaming yields

$$\exists v\, \forall x\, \big(\forall z\, \exists w\, P(c, w, z, x) \vee \forall y\, \exists w'\, \neg Q(w', y, f(x), v)\big)$$

Skolemization starts with the *outermost* existential quantifiers. First, v is replaced by a *new* constant b. We obtain

$$\forall x\, \big(\forall z\, \exists w\, P(c, w, z, x) \vee \forall y\, \exists w'\, \neg Q(w', y, f(x), b)\big)$$

Then w and w' are replaced by *new* functions g (applied to the free variables x and z) and g' (applied to the free variables x and y). We get

$$\forall x\, \big(\forall z\, P(c, g(x, z), z, x) \vee \forall y\, \neg Q(g'(x, y), y, f(x), b)\big)$$

Finally, the universal quantifiers are pushed upward. We obtain

$$\forall x\, \forall z\, \forall y\, \big(P(c, g(x, z), z, x) \vee \neg Q(g'(x, y), y, f(x), b)\big)$$

which is in CNF.

Grading scheme: Miniscoping: 4 points; Skolemization: 4 points; Rest: 4 points.

Assignment 6

Since $P(b) \prec_L \neg P(b)$, (1) implies $\neg P(c) \succ_L \neg P(b)$ and thus $P(c) \succ P(b)$.

From (2) we conclude $P(b) \succ_L R$ and thus $P(b) \succ R$.

Since we already know that $\neg P(b) \prec_L P(c)$, (3) implies $Q \succ_L P(c)$ and thus $Q \succ P(c)$.

Combining all properties, we obtain $Q \succ P(c) \succ P(b) \succ R$.

Grading scheme: 3rd, 4th, 5th, 6th correctly ordered pair of atoms: 3 points each; ignoring the fixed relationship between $\succ$ and $\succ_L$: -50%.