Automated Reasoning I, 2015
Midterm Exam, Sample Solution

Assignment 1

Suppose that S and S' are finite multisets over a set M, and that $S \succ_{\text{mul}} S'$ holds for *every* strict partial ordering $\succ$ over M. The empty relation $\succ_0$, for which $x \succ_0 y$ is false for all elements x and y, is a strict partial ordering (it is trivially irreflexive and transitive). So the property holds in particular for $\succ_0$. By the definition of the multiset extension, $S\ (\succ_0)_{\text{mul}}\ S'$ if and only if there are multisets X and Y such that $\emptyset \neq X \subseteq S$ and $S' = (S - X) \cup Y$ and for every $y \in Y$ there is an $x \in X$ such that $x \succ_0 y$. Since $x \succ_0 y$ is false for all x and y, Y must be empty. So S' equals $S - X$, this is a subset of S, and since X is non-empty, we obtain $S' \subset S$.

Notes:

- S and S' are *multisets*, not sets. So $S' \subseteq S$ means "for all $m \in M$, $S'(m) \leq S(m)$". This is not the same as "for all $m \in M$, $m \in S' \Rightarrow m \in S$", or in other words, "for all $m \in M$, $S'(m) > 0 \Rightarrow S(m) > 0$".
- One has to show $S' \neq S$ *and* $S' \subseteq S$. Proving just the first part (which is trivial by Thm. 1.10) is not sufficient.
- The assignment does *not* ask to prove the reverse direction, that is, "if $S' \subset S$ then $S \succ_{\text{mul}} S'$" (which is again obvious).

Assignment 2

Part (a) Proof: Suppose that $H[F]_p$ and $H[G]_p$ are valid. Let $\mathcal{A}$ be any valuation. By assumption, $\mathcal{A}(H[F]_p) = \mathcal{A}(H[G]_p) = 1$. If $\mathcal{A}(F) = 1$, then $\mathcal{A}(F \vee G) = \mathcal{A}(F)$, therefore, by Prop. 2.8, $\mathcal{A}(H[F \vee G]_p) = \mathcal{A}(H[F]_p) = 1$. Otherwise $\mathcal{A}(F) = 0$, then $\mathcal{A}(F \vee G) = \mathcal{A}(G)$, therefore. by Prop. 2.8, $\mathcal{A}(H[F \vee G]_p) = \mathcal{A}(H[G]_p) = 1$. So $\mathcal{A}(H[F \vee G]_p) = 1$ for every valuation $\mathcal{A}$.

Notes:

- A case analysis based on whether the validity of $H[F]_p$ depends on F or not is not useful, since the second case is just as complicated as the original problem.
- It is unavoidable to look at individual valuations $\mathcal{A}$ in the proof. One cannot replace this by a case analysis based on whether F is valid, satisfiable, or unsatisfiable.

Part (b) Counterexample: Let $F = P$ and $G = \neg P$. Then $H[F \wedge G]_1 = \neg(F \wedge G) = \neg(P \wedge \neg P)$ is valid, but $H[F]_1 = \neg F = \neg P$ and $H[G]_1 = \neg G = \neg\neg P$ are not valid.

Part (c) Proof: Suppose that $H[F]_p$ is valid and that $\text{pol}(H, p) = -1$. Let $\mathcal{A}$ be any valuation. By assumption, $\mathcal{A}(H[F]_p) = 1$. Obviously $\mathcal{A}(F \wedge G) = \min(\mathcal{A}(F), \mathcal{A}(G)) \leq \mathcal{A}(F)$, therefore, by Prop. 2.13, $\mathcal{A}(H[F \wedge G]_p) \geq \mathcal{A}(H[F]_p) = 1$. So $\mathcal{A}(H[F \wedge G]_p) = 1$ for every valuation $\mathcal{A}$.

Assignment 3

Part (a) With the given strategy, the CDCL procedure yields

$$P^{\text{d}}\ Q^{\text{d}}\ \underset{(8)}{S}\ \underset{(6)}{\neg T}\ \underset{(7)}{\neg U}\ R^{\text{d}}\ V^{\text{d}} \parallel N$$

Since all literals are defined and all clauses in N are true, this is a final state, so by Thm. 2.18, we have computed a (total) model of N.

Note:

- After $\neg U$ has been added, all clauses are true, but some literals are still undefined, so this is a partial model. The assignment asked for a total model, though.

Part (b) We use the fact that $N \models P \vee Q$ if and only if $N \cup \{\neg(P \vee Q)\}$ is unsatisfiable. In order to use the CDCL prodedure, we transform $N \cup \{\neg(P \vee Q)\}$ into a set of clauses and obtain the new clauses $\neg P$ (9) and $\neg Q$ (10). With the given strategy, the CDCL procedure yields

$$\underset{(9)}{\neg P}\ \underset{(10)}{\neg Q}\ R^{\text{d}}\ S^{\text{d}}\ \underset{(6)}{\neg T}\ \underset{(7)}{\neg U} \parallel N \cup \{(9), (10)\}$$

At this point, clause (5) is a conflict clause. By resolving (5) and (7), we obtain $Q \vee \neg S \vee T$ (which is not a backjump clause), and by resolving $Q \vee \neg S \vee T$ and (6) we obtain $Q \vee \neg S$ (11), which is a backjump clause. The best possible successor state for this backjump clause is $\neg P\ \neg Q\ \neg S \parallel N \cup \{(9), (10)\}$. After learning clause (11), we continue and obtain

$$\underset{(9)}{\neg P}\ \underset{(10)}{\neg Q}\ \underset{(11)}{\neg S}\ \underset{(3)}{V}\ \underset{(4)}{\neg U}\ \underset{(1)}{R} \parallel N \cup \{(9), (10), (11)\}$$

Now clause (2) is a conflict clause. Since there are no more decision literals, we can derive *fail*, so the clause set is unsatisfiable.

Assignment 4

Part (a) We have to show that $\succ$ is irreflexive and transitive. Irreflexivity is obvious, since $F \succ F$ implies $F \models F$ and $F \not\models F$, which is clearly a contradiction. To prove transitivity assume that $F \succ G$ and $G \succ H$, so $F \models G$, $G \models H$, $G \not\models F$, and $H \not\models G$. As shown in Exercise 2.3, $\models$ is transitive, therefore $F \models G$ and $G \models H$ imply $F \models H$. Now suppose that $H \models F$, then $F \models G$ implies $H \models G$, contradicting the assumption. Consequently. $H \not\models F$, and thus $F \succ H$.

Part (b) If Π is finite, then there are only $2^{|\Pi|}$ Π-valuations, so the set of all valuations is also finite. Now observe that $F \succ G$ implies that every valuation that is a model of F is also a model of G, but that there is at least one model of G that is not a model of F. If there is a chain $F_1 \succ F_2 \succ F_3 \succ \dots$, then the number of models grows in each step, but this number is bounded by $2^{|\Pi|}$. So the chain cannot be infinite.

Notes:

- $F \succ G$ is equivalent to "$\forall \mathcal{A}$: $\mathcal{A}(F) \leq \mathcal{A}(G)$ and $\exists \mathcal{A}'$: $\mathcal{A}'(F) < \mathcal{A}'(G)$." Ignoring the quantifications leads to non-sensical results.
- The elements of the chain are formulas over Π, not necessarily elements of Π.
- Even if Π is finite, there are infinitely many Π-formulas. The set of equivalence classes of formulas is finite, though; this can be proved either by looking at the sets of models (as above), or using the fact that every Π-formula is equivalent to some formula in CNF without duplicated literals or clauses.

Part (c) If $\Pi = \{P_1, P_2, P_3, \dots\}$ is infinite, define $F_i = \bigvee_{1 \leq j \leq i} P_j$, then $F_1 \succ F_2 \succ F_3 \succ \dots$ is an infinite descending chain.

Note:

- There is no infinite descending chain whose elements are only propositional variables from Π, since for any two different propositional variables P and Q we always have $P \not\models Q$ and therefore $P \not\succ Q$.

Assignment 5

The Σ-algebra $\mathcal{A}$ with $U_\mathcal{A} = \{2, 3\}$, $b_\mathcal{A} = 2$, $c_\mathcal{A} = 2$, $d_\mathcal{A} = 3$, $f_\mathcal{A}(u) = 3$ for all $u \in U_\mathcal{A}$, and $P_\mathcal{A} = \{2\}$ is a model of the given formula; its universe has two elements.

Assignment 6

We first compute the negation normal form of F, namely

$$\forall x\, \exists y\, \Big(\big(\neg P(b) \wedge \exists z\, \neg Q(y, z) \big) \vee R(x, y) \Big)$$

Miniscoping yields

$$\big(\neg P(b) \wedge \exists y\, \exists z\, \neg Q(y, z) \big) \vee \forall x\, \exists y\, R(x, y)$$

and variable renaming yields

$$\big(\neg P(b) \wedge \exists y\, \exists z\, \neg Q(y, z) \big) \vee \forall x\, \exists y'\, R(x, y')$$

By Skolemization we obtain

$$\big(\neg P(b) \wedge \neg Q(c, d) \big) \vee \forall x\, R(x, f(x))$$

with Skolem functions $c/0$, $d/0$, and $f/1$. Finally, we push $\forall$ upward and apply the distributivity law to get the conjunctive normal form

$$\forall x\, \Big(\big(\neg P(b) \vee R(x, f(x)) \big) \wedge \big(\neg Q(c, d) \vee R(x, f(x)) \big) \Big)$$

Notes:

- Skolemization starts with the *outermost* existential quantifiers.
- Every Skolem function symbol that is introduced must be *new*, that is, different from all symbols from Σ and all previously introduced Skolem function symbols.