

4.6 Knuth-Bendix Completion

Completion:

Goal: Given a set E of equations, transform E into an equivalent convergent set R of rewrite rules.

(If R is finite: decision procedure for E .)

Knuth-Bendix Completion: Idea

How to ensure termination?

Fix a reduction ordering \succ and construct R in such a way that $\rightarrow_R \subseteq \succ$ (i. e., $l \succ r$ for every $l \rightarrow r \in R$).

How to ensure confluence?

Check that all critical pairs are joinable.

Note: Every critical pair $\langle s, t \rangle$ can be *made* joinable by adding $s \rightarrow t$ or $t \rightarrow s$ to R .

(Actually, we first add $s \approx t$ to E and later try to turn it into a rule that is contained in \succ ; this gives us some additional degree of freedom.)

Knuth-Bendix Completion: Inference Rules

The completion procedure is presented as a set of inference rules working on a set of equations E and a set of rules R : $E_0, R_0 \vdash E_1, R_1 \vdash E_2, R_2 \vdash \dots$

At the beginning, $E = E_0$ is the input set and $R = R_0$ is empty. At the end, E should be empty; then R is the result.

For each step $E, R \vdash E', R'$, the equational theories of $E \cup R$ and $E' \cup R'$ agree: $\approx_{E \cup R} = \approx_{E' \cup R'}$.

Notations:

The formula $s \dot{\approx} t$ denotes either $s \approx t$ or $t \approx s$.

$CP(R)$ denotes the set of all critical pairs between rules in R .

Orient:

$$\frac{E \cup \{s \approx t\}, R}{E, R \cup \{s \rightarrow t\}} \quad \text{if } s \succ t$$

Note: There are equations $s \approx t$ that cannot be oriented, i. e., neither $s \succ t$ nor $t \succ s$.

Trivial equations cannot be oriented – but we don't need them anyway:

Delete:

$$\frac{E \cup \{s \approx s\}, R}{E, R}$$

Critical pairs between rules in R are turned into additional equations:

Deduce:

$$\frac{E, R}{E \cup \{s \approx t\}, R} \quad \text{if } \langle s, t \rangle \in \text{CP}(R).$$

Note: If $\langle s, t \rangle \in \text{CP}(R)$ then $s \leftarrow_R u \rightarrow_R t$ and hence $R \models s \approx t$.

The following inference rules are not absolutely necessary, but very useful (e. g., to get rid of joinable critical pairs and to deal with equations that cannot be oriented):

Simplify-Eq:

$$\frac{E \cup \{s \approx t\}, R}{E \cup \{u \approx t\}, R} \quad \text{if } s \rightarrow_R u.$$

Simplification of the right-hand side of a rule is unproblematic:

R-Simplify-Rule:

$$\frac{E, R \cup \{s \rightarrow t\}}{E, R \cup \{s \rightarrow u\}} \quad \text{if } t \rightarrow_R u.$$

Simplification of the left-hand side may influence orientability and orientation. Therefore, it yields an *equation*:

L-Simplify-Rule:

$$\frac{E, R \cup \{s \rightarrow t\}}{E \cup \{u \approx t\}, R} \quad \text{if } s \rightarrow_R u \text{ using a rule } l \rightarrow r \in R \text{ such that } s \sqsupset l \text{ (see below).}$$

For technical reasons, the lhs of $s \rightarrow t$ may only be simplified using a rule $l \rightarrow r$, if $l \rightarrow r$ cannot be simplified using $s \rightarrow t$, that is, if $s \sqsupset l$, where the *encompassment quasi-ordering* \sqsupseteq is defined by

$$s \sqsupseteq l \text{ if } s|_p = l\sigma \text{ for some } p \text{ and } \sigma$$

and $\sqsupset = \sqsupseteq \setminus \sqsubseteq$ is the strict part of \sqsupseteq .

Lemma 4.32 \sqsupset is a well-founded strict partial ordering.

Lemma 4.33 If $E, R \vdash E', R'$, then $\approx_{E \cup R} = \approx_{E' \cup R'}$.

Lemma 4.34 If $E, R \vdash E', R'$ and $\rightarrow_R \subseteq \succ$, then $\rightarrow_{R'} \subseteq \succ$.

Note: Like in ordered resolution, simplification should be preferred to deduction:

- Simplify/delete whenever possible.
- Otherwise, orient an equation, if possible.
- Last resort: compute critical pairs.

Knuth-Bendix Completion: Correctness Proof

If we run the completion procedure on a set E of equations, different things can happen:

- (1) We reach a state where no more inference rules are applicable and E is not empty.
 \Rightarrow Failure (try again with another ordering?)
- (2) We reach a state where E is empty and all critical pairs between the rules in the current R have been checked.
- (3) The procedure runs forever.

In order to treat these cases simultaneously, we need some definitions.

A (finite or infinite sequence) $E_0, R_0 \vdash E_1, R_1 \vdash E_2, R_2 \vdash \dots$ with $R_0 = \emptyset$ is called a *run* of the completion procedure with input E_0 and \succ .

For a run, $E_\infty = \bigcup_{i \geq 0} E_i$ and $R_\infty = \bigcup_{i \geq 0} R_i$.

The sets of *persistent equations or rules* of the run are $E_* = \bigcup_{i \geq 0} \bigcap_{j \geq i} E_j$ and $R_* = \bigcup_{i \geq 0} \bigcap_{j \geq i} R_j$.

Note: If the run is finite and ends with E_n, R_n , then $E_* = E_n$ and $R_* = R_n$.

A run is called *fair*, if $CP(R_*) \subseteq E_\infty$ (i. e., if every critical pair between persisting rules is computed at some step of the derivation).

Goal:

Show: If a run is fair and E_* is empty, then R_* is convergent and equivalent to E_0 .

In particular: If a run is fair and E_* is empty, then $\approx_{E_0} = \approx_{E_\infty \cup R_\infty} = \leftrightarrow_{E_\infty \cup R_\infty}^* = \downarrow_{R_*}$.

General assumptions from now on:

$E_0, R_0 \vdash E_1, R_1 \vdash E_2, R_2 \vdash \dots$ is a fair run.

R_0 and E_* are empty.

A *proof* of $s \approx t$ in $E_\infty \cup R_\infty$ is a finite sequence (s_0, \dots, s_n) such that $s = s_0$, $t = s_n$, and for all $i \in \{1, \dots, n\}$:

- (1) $s_{i-1} \leftrightarrow_{E_\infty} s_i$, or
- (2) $s_{i-1} \rightarrow_{R_\infty} s_i$, or
- (3) $s_{i-1} \leftarrow_{R_\infty} s_i$.

The pairs (s_{i-1}, s_i) are called *proof steps*.

A proof is called a *rewrite proof in R_** , if there is a $k \in \{0, \dots, n\}$ such that $s_{i-1} \rightarrow_{R_*} s_i$ for $1 \leq i \leq k$ and $s_{i-1} \leftarrow_{R_*} s_i$ for $k+1 \leq i \leq n$.

Idea (Bachmair, Dershowitz, Hsiang):

Define a well-founded ordering on proofs, such that for every proof that is not a rewrite proof in R_* there is an equivalent smaller proof.

Consequence: For every proof there is an equivalent rewrite proof in R_* .

We associate a *cost* $c(s_{i-1}, s_i)$ with every proof step as follows:

- (1) If $s_{i-1} \leftrightarrow_{E_\infty} s_i$, then $c(s_{i-1}, s_i) = (\{s_{i-1}, s_i\}, -, -)$, where the first component is a multiset of terms and $-$ denotes an arbitrary (irrelevant) term.
- (2) If $s_{i-1} \rightarrow_{R_\infty} s_i$ using $l \rightarrow r$, then $c(s_{i-1}, s_i) = (\{s_{i-1}\}, l, s_i)$.
- (3) If $s_{i-1} \leftarrow_{R_\infty} s_i$ using $l \rightarrow r$, then $c(s_{i-1}, s_i) = (\{s_i\}, l, s_{i-1})$.

Proof steps are compared using the lexicographic combination of the multiset extension of the reduction ordering \succ , the encompassment ordering \sqsupseteq , and the reduction ordering \succ .

The cost $c(P)$ of a proof P is the multiset of the costs of its proof steps.

The *proof ordering* \succ_C compares the costs of proofs using the multiset extension of the proof step ordering.

Lemma 4.35 \succ_C is a well-founded ordering.

Lemma 4.36 Let P be a proof in $E_\infty \cup R_\infty$. If P is not a rewrite proof in R_* , then there exists an equivalent proof P' in $E_\infty \cup R_\infty$ such that $P \succ_C P'$.

Proof. If P is not a rewrite proof in R_* , then it contains

- (a) a proof step that is in E_∞ , or
- (b) a proof step that is in $R_\infty \setminus R_*$, or
- (c) a subproof $s_{i-1} \leftarrow_{R_*} s_i \rightarrow_{R_*} s_{i+1}$ (peak).

We show that in all three cases the proof step or subproof can be replaced by a smaller subproof:

Case (a): A proof step using an equation $s \dot{\approx} t$ is in E_∞ . This equation must be deleted during the run.

If $s \dot{\approx} t$ is deleted using *Orient*:

$$\dots s_{i-1} \leftrightarrow_{E_\infty} s_i \dots \implies \dots s_{i-1} \rightarrow_{R_\infty} s_i \dots$$

If $s \dot{\approx} t$ is deleted using *Delete*:

$$\dots s_{i-1} \leftrightarrow_{E_\infty} s_{i-1} \dots \implies \dots s_{i-1} \dots$$

If $s \dot{\approx} t$ is deleted using *Simplify-Eq*:

$$\dots s_{i-1} \leftrightarrow_{E_\infty} s_i \dots \implies \dots s_{i-1} \rightarrow_{R_\infty} s' \leftrightarrow_{E_\infty} s_i \dots$$

Case (b): A proof step using a rule $s \rightarrow t$ is in $R_\infty \setminus R_*$. This rule must be deleted during the run.

If $s \rightarrow t$ is deleted using *R-Simplify-Rule*:

$$\dots s_{i-1} \rightarrow_{R_\infty} s_i \dots \implies \dots s_{i-1} \rightarrow_{R_\infty} s' \leftarrow_{R_\infty} s_i \dots$$

If $s \rightarrow t$ is deleted using *L-Simplify-Rule*:

$$\dots s_{i-1} \rightarrow_{R_\infty} s_i \dots \implies \dots s_{i-1} \rightarrow_{R_\infty} s' \leftrightarrow_{E_\infty} s_i \dots$$

Case (c): A subproof has the form $s_{i-1} \leftarrow_{R_*} s_i \rightarrow_{R_*} s_{i+1}$.

If there is no overlap or a non-critical overlap:

$$\dots s_{i-1} \leftarrow_{R_*} s_i \rightarrow_{R_*} s_{i+1} \dots \implies \dots s_{i-1} \rightarrow_{R_*}^* s' \leftarrow_{R_*}^* s_{i+1} \dots$$

If there is a critical pair that has been added using *Deduce*:

$$\dots s_{i-1} \leftarrow_{R_*} s_i \rightarrow_{R_*} s_{i+1} \dots \implies \dots s_{i-1} \leftrightarrow_{E_\infty} s_{i+1} \dots$$

In all cases, checking that the replacement subproof is smaller than the replaced subproof is routine. \square

Theorem 4.37 Let $E_0, R_0 \vdash E_1, R_1 \vdash E_2, R_2 \vdash \dots$ be a fair run and let R_0 and E_* be empty. Then

- (1) every proof in $E_\infty \cup R_\infty$ is equivalent to a rewrite proof in R_* ,
- (2) R_* is equivalent to E_0 , and
- (3) R_* is convergent.

Proof. (1) By well-founded induction on \succ_C using the previous lemma.

(2) Clearly $\approx_{E_\infty \cup R_\infty} = \approx_{E_0}$. Since $R_* \subseteq R_\infty$, we get $\approx_{R_*} \subseteq \approx_{E_\infty \cup R_\infty}$. On the other hand, by (1), $\approx_{E_\infty \cup R_\infty} \subseteq \approx_{R_*}$.

(3) Since $\rightarrow_{R_*} \subseteq \succ$, R_* is terminating. By (1), R_* is confluent. □

4.7 Unfailing Completion

Classical completion:

Try to transform a set E of equations into an equivalent convergent TRS.

Fail, if an equation can neither be oriented nor deleted.

Unfailing completion (Bachmair, Dershowitz and Plaisted):

If an equation cannot be oriented, we can still use *orientable instances* for rewriting.

Note: If \succ is total on ground terms, then every *ground instance* of an equation is trivial or can be oriented.

Goal: Derive a *ground convergent* set of equations.

Let E be a set of equations, let \succ be a reduction ordering.

We define the relation \rightarrow_{E^\succ} by

$$s \rightarrow_{E^\succ} t \quad \text{iff} \quad \begin{array}{l} \text{there exist } (u \approx v) \in E \text{ or } (v \approx u) \in E, \\ p \in \text{pos}(s), \text{ and } \sigma : X \rightarrow T_\Sigma(X), \\ \text{such that } s|_p = u\sigma \text{ and } t = s[v\sigma]_p \text{ and } u\sigma \succ v\sigma. \end{array}$$

Note: \rightarrow_{E^\succ} is terminating by construction.

From now on let \succ be a reduction ordering that is total on ground terms.

E is called *ground convergent* w. r. t. \succ , if for all ground terms s and t with $s \leftrightarrow_E^* t$ there exists a ground term v such that $s \rightarrow_{E^\succ}^* v \leftarrow_{E^\succ}^* t$. (Analogously for $E \cup R$.)

As for standard completion, we establish ground convergence by computing critical pairs.

However, the ordering \succ is not total on non-ground terms. Since $s\theta \succ t\theta$ implies $s \not\prec t$, we approximate \succ on ground terms by $\not\prec$ on arbitrary terms.

Let $u_i \approx v_i$ ($i = 1, 2$) be equations in E whose variables have been renamed such that $\text{var}(u_1 \approx v_1) \cap \text{var}(u_2 \approx v_2) = \emptyset$. Let $p \in \text{pos}(u_1)$ be a position such that $u_1|_p$ is not a variable, σ is an mgu of $u_1|_p$ and u_2 , and $u_i\sigma \not\prec v_i\sigma$ ($i = 1, 2$). Then $\langle v_1\sigma, (u_1\sigma)[v_2\sigma]_p \rangle$ is called a *semi-critical pair* of E with respect to \succ .

The set of all semi-critical pairs of E is denoted by $\text{SP}_\succ(E)$.

Semi-critical pairs of $E \cup R$ are defined analogously. If $\rightarrow_R \subseteq \succ$, then $\text{CP}(R)$ and $\text{SP}_\succ(R)$ agree.

Note: In contrast to critical pairs, it may be necessary to consider overlaps of an equation with itself at the top. For instance, if $E = \{f(x) \approx g(y)\}$, then $\langle g(y), g(y') \rangle$ is a non-trivial semi-critical pair.

The *Deduce* rule takes now the following form:

Deduce:

$$\frac{E, R}{E \cup \{s \approx t\}, R} \quad \text{if } \langle s, t \rangle \in \text{SP}_\succ(E \cup R).$$

Moreover, the fairness criterion for runs is replaced by

$$\text{SP}_\succ(E_* \cup R_*) \subseteq E_\infty$$

(i. e., if every semi-critical pair between persisting rules or equations is computed at some step of the derivation).

Analogously to Thm. 4.37 we obtain now the following theorem:

Theorem 4.38 *Let $E_0, R_0 \vdash E_1, R_1 \vdash E_2, R_2 \vdash \dots$ be a fair run; let $R_0 = \emptyset$. Then*

- (1) $E_* \cup R_*$ is equivalent to E_0 , and
- (2) $E_* \cup R_*$ is ground convergent.

Moreover one can show that, whenever there exists a *reduced* convergent R such that $\approx_{E_0} = \downarrow_R$ and $\rightarrow_R \in \succ$, then for every fair and *simplifying* run $E_* = \emptyset$ and $R_* = R$ up to variable renaming.

Here R is called *reduced*, if for every $l \rightarrow r \in R$, both l and r are irreducible w. r. t. $R \setminus \{l \rightarrow r\}$. A run is called *simplifying*, if R_* is reduced, and for all equations $u \approx v \in E_*$, u and v are incomparable w. r. t. \succ and irreducible w. r. t. R_* .

Unfailing completion is refutationally complete for equational theories:

Theorem 4.39 *Let E be a set of equations, let \succ be a reduction ordering that is total on ground terms. For any two terms s and t , let \hat{s} and \hat{t} be the terms obtained from s and t by replacing all variables by Skolem constants. Let $eq/2$, $true/0$ and $false/0$ be new operator symbols, such that $true$ and $false$ are smaller than all other terms. Let $E_0 = E \cup \{eq(\hat{s}, \hat{t}) \approx true, eq(x, x) \approx false\}$. If $E_0, \emptyset \vdash E_1, R_1 \vdash E_2, R_2 \vdash \dots$ be a fair run of unfailing completion, then $s \approx_E t$ iff some $E_i \cup R_i$ contains $true \approx false$.*

Outlook:

Combine ordered resolution and unfailing completion to get a calculus for equational clauses:

compute inferences between (strictly) maximal literals as in ordered resolution,
 compute overlaps between maximal sides of equations as in unfailing completion

\Rightarrow Superposition calculus.