Assignment 1 *(Well-founded orderings)* (12 + 5 = 17 points)

Part (a) Prove: If $>$ is a well-founded strict partial ordering on a set M and if b is the only element of M that is minimal in M, then b is the smallest element of M.

Part (b) Give an example of a strict partial ordering $>$ on a set M and an element $b \in M$ such that b is the only element of M that is minimal in M but not the smallest element of M.

Assignment 2 *(Algebras)* (12 points)

Let $\Sigma = (\Omega, \Pi)$ be a first-order signature with $\Omega = \{b/0,\ f/1\}$ and $\Pi = \{P/1\}$. Let F be the Σ-formula

$$\neg P(b)\ \wedge\ P(f(f(b)))\ \wedge\ \forall x\,(P(x) \vee P(f(x))).$$

Decide for each of the following statements whether they are true or false:

(1) If $\mathcal{A}$ is a Σ-model of F, then $P_{\mathcal{A}} \neq \emptyset$ and $P_{\mathcal{A}} \neq U_{\mathcal{A}}$.

(2) There is a Σ-model $\mathcal{A}$ of F such that $U_{\mathcal{A}} = \{7, 8, 9\}$.

(3) There is a Σ-model $\mathcal{A}$ of F such that $f_{\mathcal{A}}(a) = f_{\mathcal{A}}(a')$ for all $a, a' \in U_{\mathcal{A}}$.

(4) F has exactly four Σ-models.

(5) There are infinitely many Herbrand interpretations over Σ.

(6) There is an Herbrand model of F over Σ with a finite universe.

(7) There is an Herbrand model $\mathcal{A}$ of F over Σ and an assignment β such that $\mathcal{A}(\beta)(f(b)) = \mathcal{A}(\beta)(f(f(b)))$.

(Note on grading: A yes/no answer is sufficient; you do not have to give an explanation. However, you need at least four correct answers to get any points for this assignment. Missing answers count like false answers.)

Assignment 3 *(Critical pairs)* (12 points)

Let R be the following set of rewrite rules over $\Sigma = (\{f/1,\ g/2,\ h/1,\ c/0\}, \emptyset)$.

$$f(f(x)) \to h(h(x)) \qquad (1)$$

$$g(f(y), x) \to g(y, x) \qquad (2)$$

$$h(g(z, f(c))) \to f(z) \qquad (3)$$

Give all (non-trivial) critical pairs between the three rules.

Assignment 4 *(Confluence)* (12 points)

Let $\Sigma = (\Omega, \emptyset)$ be a finite signature, let $\succ$ be a simplification ordering. Let R be a TRS over $\mathrm{T}_\Sigma(X)$ such that $l \succ r$ for all $l \to r \in R$. Let h be an n-ary function symbol in Ω (with $n > 0$) that does not occur in any left-hand side of a rule in R. Prove: If R is confluent, then $R \cup \{\, h(x, \ldots, x) \to x \,\}$ is confluent.

Assignment 5 *(Reduction orderings)* (5 + 5 + 5 = 15 points)

Part (a) Let $\Sigma = (\{f/2,\ g/2,\ h/2\}, \emptyset)$; let R be the term rewrite system

$$\{\ g(x, f(x,y)) \to h(y, g(x,y)), \quad h(x,y) \to g(y,y)\ \}$$

Is there a lexicographic path ordering $\succ_{\text{lpo}}$ such that $\to_R \subseteq \succ_{\text{lpo}}$? If yes, give the precedence of this LPO; if no, explain why such an LPO does not exist.

Part (b) Let $\Sigma = (\{f/2,\ g/1,\ h/1, b/0\}, \emptyset)$; let R be the term rewrite system

$$\{\ f(g(x), y) \to g(f(x,x)), \quad h(f(x,b)) \to g(x)\ \}$$

Is there a Knuth-Bendix ordering $\succ_{\text{kbo}}$ such that $\to_R \subseteq \succ_{\text{kbo}}$? If yes, give the weights and precedence of this KBO; if no, explain why such a KBO does not exist.

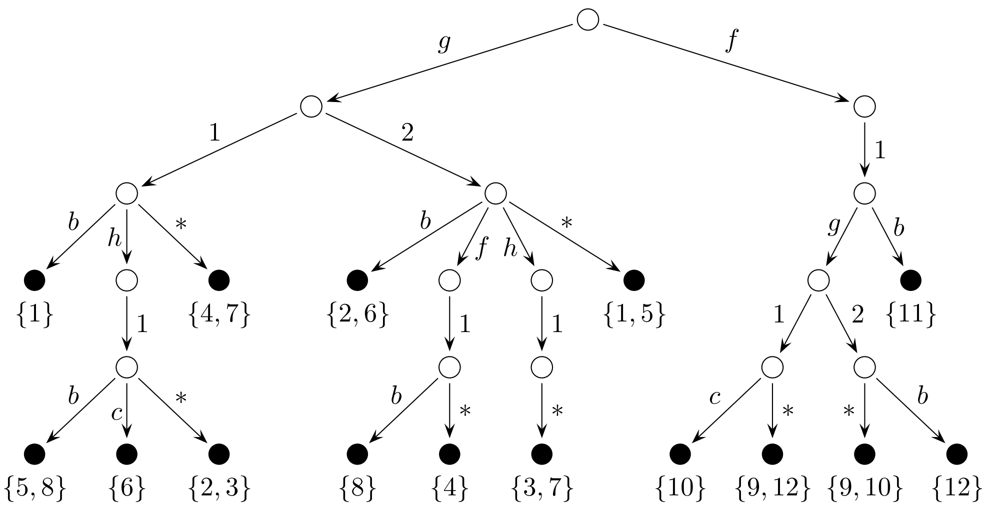
Part (c) Let $\Sigma = (\{f/1,\ g/1,\ b/0,\ c/0\}, \emptyset)$; let R be the term rewrite system

$$\{\ f(g(x)) \to g(g(f(x))), \quad c \to f(b)\ \}$$

Is there a polynomial ordering $\succ_{\mathcal{A}}$ in which the function symbols are interpreted by linear polynomials over $U_{\mathcal{A}} = \{\, n \in \mathbb{N} \mid n \geq 1 \,\}$ such that $\to_R \subseteq \succ_{\mathcal{A}}$? If yes, give the polynomials by which the symols of Σ are interpreted; if no, explain why such an ordering does not exist.

Assignment 6 *(Path indexing)* (3 + 3 + 6 = 12 points)

Consider the following path index:



Part (a) Which terms have the numbers 3, 5, and 12 in the path index?

Part (b) Which of the terms $g(*, h(*))$, $f(g(c,b))$, and $g(h(*), b)$ are contained in the path index? If they are contained, what are their numbers?

Part (c) Assume that the terms in the path index are the left-hand sides of the rewrite rules of a TRS R. Is the term $f(g(h(c), f(b)))$ reducible by rules in R? If yes, what are the numbers of the left-hand sides of these rules?