Assignment 1 *(Semantics of FOL)* (14 points)

Let $\Sigma = (\Omega, \Pi)$ be a signature. For every Σ-formula F without equality let $\mathrm{neg}(F)$ be the formula that one obtains from F by replacing every atom $P(t_1, \dots, t_n)$ in F by its negation $\neg P(t_1, \dots, t_n)$ for every $P/n \in \Pi$. Prove: If F is valid, then $\mathrm{neg}(F)$ is valid.

(Note: Somewhere in the proof you need an induction over the structure of formulas. It is sufficient if you check the base cases and $\wedge$, $\neg$, and $\exists$. The other boolean connectives and quantifiers ($\vee$, $\rightarrow$, $\leftrightarrow$, $\forall$) can be handled analogously; you may omit them.)

Assignment 2 *(Resolution)* (8 + 6 = 14 points)

Let $\Sigma = (\{f/1,\ g/1,\ h/1,\ b/0,\ c/0\},\ \{P/2,\ Q/1,\ R/2\})$; let N be the following set of clauses over Σ:

$$P(g(x), x) \vee P(b, x) \vee R(f(x), x) \qquad (1)$$

$$\neg P(g(x), g(x)) \qquad (2)$$

$$\neg P(z, h(y)) \vee \neg R(y, z) \qquad (3)$$

$$\neg P(y, c) \vee \neg P(z, b) \vee \neg Q(z) \vee R(z, y) \qquad (4)$$

$$Q(b) \vee Q(x) \vee \neg R(f(x), x) \qquad (5)$$

Part (a) Suppose that the atom ordering $\succ$ is a lexicographic path ordering with the precedence $P > Q > R > f > g > h > b > c$ and that the selection function *sel* selects no literals. Compute all $\mathrm{Res}^{\succ}_{sel}$ inferences between the clauses (1)–(5). (Do not compute inferences between derived clauses. Do not compute any inferences that violate the restrictions of the calculus.)

Part (b) If the selection function *sel* is defined appropriately, the set N is saturated under $\mathrm{Res}^{\succ}_{sel}$ (with $\succ$ as in Part (a)). Which literals have to be selected?

Assignment 3 *(E-Algebras)* (12 points)

Let $\Sigma = (\Omega, \emptyset)$ be a first-order signature with $\Omega = \{f/2,\ b/0,\ c/0,\ d/0\}$. Let E be the set of Σ-equations

$$\{\, \forall x\, (f(x, c) \approx b),\ \ c \approx d\, \},$$

let $X = \{x, y, z\}$ be a set of variables. For any $t \in \mathrm{T}_\Sigma(X)$ let $[t]$ denote the congruence class of t w.r.t. E. Let $\mathcal{T} = \mathrm{T}_\Sigma(X)/E$ and let $\beta : X \to U_{\mathcal{T}}$ be the assignment that maps every variable to $[c]$. Decide for each of the following statements whether they are true or false:

(1) $[c]$ is a finite set of Σ-terms.

(2) $[f(c, c)]$ is a set of ground Σ-terms.

(3) $[x]$ is an element of the universe of $\mathcal{T}$.

(4) $\{b, f(x, c)\}$ is a congruence class w.r.t. E.

(5) $f(c, b) \in [f(d, b)]$.

(6) $f_{\mathcal{T}}([y], [d]) = [f(z, c)]$.

(7) $\mathcal{T}(\beta)(y \approx d) = 1$.

(8) $\mathcal{T}(\beta)(\forall z\, (z \approx c)) = 1$.

(Note on grading: A yes/no answer is sufficient; you do not have to give any explanations. However, you need at least five correct answers to get any points for assignment 3. Missing answers count like false answers.)

Assignment 4 *(Rewriting)* (12 points)

Let $\Sigma = (\Omega, \emptyset)$ with $\Omega = \{f/1, g/1, h/1, b/0, c/0\}$. Let R be the following term rewrite system over Σ:

$$\{\, g(f(x)) \to h(x),\ h(f(x)) \to g(x),\ g(b) \to c,\ h(c) \to b \,\}$$

Prove: If $s, t \in \mathrm{T}_\Sigma(X)$ and $R \models \forall \vec{x}\,(s \approx t)$, then there exists a rewrite derivation $s \leftrightarrow^*_R t$ with at most $|s| + |t| - 2$ rewrite steps.

Assignment 5 *(Reduction Orderings)* (12 points)

Let $\Sigma = (\Omega, \emptyset)$ be a finite signature. For $t \in \mathrm{T}_\Sigma(X)$ we define $\mathrm{depth}(t) = \max\{\, |p| \mid p \in \mathrm{pos}(t) \,\}$. Let $\succ$ be a strict partial ordering on Ω. The binary relation $\succ_{\mathrm{do}}$ on $\mathrm{T}_\Sigma(X)$ is defined by: $s \succ_{\mathrm{do}} t$ if and only if

(1) $\#(x, s) \geq \#(x, t)$ for all variables x and $\mathrm{depth}(s) > \mathrm{depth}(t)$, or

(2) $\#(x, s) \geq \#(x, t)$ for all variables x, $\mathrm{depth}(s) = \mathrm{depth}(t)$, and

  (a) $s = f(s_1, \dots, s_m)$, $t = g(t_1, \dots, t_n)$, and $f \succ g$, or
  (b) $s = f(s_1, \dots, s_m)$, $t = f(t_1, \dots, t_m)$, and $(s_1, \dots, s_m)\ (\succ_{\mathrm{do}})_{\mathrm{lex}}\ (t_1, \dots, t_m)$.

Give an example that shows that $>_{\mathrm{do}}$ is *not* a reduction ordering.

Assignment 6 *(Dependency Pairs)* (4 + 8 + 4 = 16 points)

Part (a) Let $\Sigma = (\Omega, \emptyset)$ with $\Omega = \{f/2,\ g/2,\ h/1,\ k/1,\ b/0\}$. Compute the dependency pairs of the following rewrite system R over Σ:

$$f(x, h(x)) \to h(k(x)) \qquad (1)$$
$$f(h(x), y) \to g(x, g(h(x), x)) \qquad (2)$$
$$g(x, x) \to f(x, x) \qquad (3)$$
$$g(x, y) \to y \qquad (4)$$
$$h(b) \to b \qquad (5)$$

Part (b) Compute the approximated dependency graph for R (using cap and ren) and use the subterm criterion to show that R is terminating. If a graph is modified, depict both the original and the modified graph and indicate the strongly connected components in the graphs.

Part (c) The approximated dependency graph contains an edge from a dependency pair generated by rule (3) to a dependency pair generated by rule (1). Is this edge also contained in the exact dependency graph? Give an explanation.