max planck institut informatik

Universität des Saarlandes
FR Informatik

Uwe Waldmann **January 8, 2020**

Tutorials for "Automated Reasoning"
Exercise sheet 10

Exercise 10.1: (*5 P*)
Let $E = \{\, f(g(x)) \approx g(f(x)) \,\}$. Give a derivation for $E \vdash f(f(g(g(y)))) \approx g(g(f(f(y))))$.

Exercise 10.2: (*6 P*)
Let $\Sigma = (\{f/2, b/0, c/0, d/0\}, \emptyset)$; let $E = \{\forall x\, (f(x,x) \approx b),\ c \approx d\}$; let $X = \{x, y, z\}$ be a set of variables. For any $t \in \mathrm{T}_\Sigma(X)$ let $[t]$ denote the congruence class of t w.r.t. E. Let $\mathcal{T} = \mathrm{T}_\Sigma(X)/E$ and let $\beta : X \to \mathcal{T}$ be the assignment that maps every variable to $[c]$.

State for each of the following "items" whether it is (a) a term, (b) a set of terms, (c) a set of ground terms, (d) a congruence class w.r.t. E, (e) a formula, (f) a truth value, or (g) non-sensical. (More than one answer may be correct.)

(1) $[c]$
(2) $[b]$
(3) $\{d\}$
(4) $f(c,d)$
(5) $[f(c,d)]$
(6) $b \approx c$
(7) $f([c],[d])$
(8) $f_{\mathcal{T}}(\{c,d\},\{c,d\})$
(9) $\mathcal{T}(x)$
(10) $\mathcal{T}(\beta)(f(c,y))$
(11) $f_{\mathcal{T}}(\mathcal{T}(\beta)(c), \mathcal{T}(\beta)(y))$
(12) $\mathcal{T}(\beta)(\forall x, y\ (f(x,x) \approx f(y,y)))$

Exercise 10.3: (*4 P*)
Is the rewrite system

$$\{\, f(a) \to f(b),\ f(b) \to f(c),\ f(c) \to f(a),\ f(x) \to x \,\}$$

(i) terminating, (ii) normalizing, (iii) locally confluent, (iv) confluent? Give a brief explanation.

Exercise 10.4: (*5 P*)
A friend asks you to proofread her master thesis. On page 15 of the thesis, your friend writes the following:

> **Lemma 5.** Let $\succ$ be a well-founded ordering over a set A, let $\to$ be a binary relation such that $\to \subseteq \succ$. Let r be an element of A that is irreducible with respect to $\to$, and define $A_r = \{ t \in A \mid t \to^* r \}$. If for every $u_0, u_1, u_2 \in A$ such that $u_1 \leftarrow u_0 \to u_2 \to^* r$ there exists a $u_3 \in A$ such that $u_1 \to^* u_3 \leftarrow^* u_2$, then for every $t_0 \in A_r$ and $t_1 \in A$, $t_0 \to^* t_1$ implies $t_1 \in A_r$.
>
> **Proof.** We use well-founded induction over t_0 with respect to $\succ$. Let $t_0 \in A_r$ and $t_1 \in A$ such that $t_0 \to^* t_1$. If this derivation is empty, the result is trivial, so suppose that $t_0 \to t_1' \to^* t_1$. As $t_0 \in A_r$ is reducible, it is different from r, hence there is a non-empty derivation $t_0 \to t_2 \to^* r$. By assumption, there exists a $t_3 \in A$ such that $t_1' \to^* t_3 \leftarrow^* t_2$. Now $t_0 \succ t_2$ and $t_2 \in A_r$, hence $t_3 \in A_r$ by the induction hypothesis, and thus $t_1' \in A_r$. Since $t_0 \succ t_1'$, we can use the induction hypothesis once more and obtain $t_1 \in A_r$ as required.

(1) Is the "proof" correct (yes/no)?

(2) If the "proof" is not correct:

   (a) Which step is incorrect?

   (b) Does the "theorem" hold? If yes, give a correct proof, otherwise give a counterexample.

Challenge Problem: (*6 Bonus Points*)

(a) Show that the compactness theorem (Thm. 3.39) holds also for first-order logic with equality. (You may use all results proved in the lecture so far.)

(b) Use the compactness theorem for first-order logic with equality to prove the following statement: Let F be a first-order formula with equality. If, for every natural number n, F has a model whose universe has at least n elements, then F has a model with an infinite universe.

Submit your solution in lecture hall E1.3, Room 001 during the lecture on January 15. Please write the time of your tutorial group (Mon or Tue) on your solution.

Joint solutions, prepared by up to three persons together, are allowed (but not encouraged). If you prepare your solution jointly, submit it only once and indicate all authors on the sheet.