SU: Main Properties

If $E = \{x_1 \doteq u_1, \ldots, x_k \doteq u_k\}$, with x_i pairwise distinct, $x_i \notin \mathrm{var}(u_j)$, then E is called an (equational problem in) *solved form* representing the solution $\sigma_E = \{x_1 \mapsto u_1, \ldots, x_k \mapsto u_k\}$.

Proposition 3.26 *If E is a solved form then σ_E is an mgu of E.*

Theorem 3.27

1. *If $E \Rightarrow_{SU} E'$ then σ is a unifier of E iff σ is a unifier of E'*
2. *If $E \Rightarrow^*_{SU} \bot$ then E is not unifiable.*
3. *If $E \Rightarrow^*_{SU} E'$ with E' in solved form, then $\sigma_{E'}$ is an mgu of E.*

Proof. (1) We have to show this for each of the rules. Let's treat the case for the 4th rule here. Suppose σ is a unifier of $x \doteq t$, that is, $x\sigma = t\sigma$. Thus, $\sigma \circ \{x \mapsto t\} = \sigma[x \mapsto t\sigma] = \sigma[x \mapsto x\sigma] = \sigma$. Therefore, for any equation $u \doteq v$ in E: $u\sigma = v\sigma$, iff $u\{x \mapsto t\}\sigma = v\{x \mapsto t\}\sigma$. (2) and (3) follow by induction from (1) using Proposition 3.26. □

Main Unification Theorem

Theorem 3.28 *E is unifiable if and only if there is a most general unifier σ of E, such that σ is idempotent and $\mathrm{dom}(\sigma) \cup \mathrm{codom}(\sigma) \subseteq \mathrm{var}(E)$.*

Proof. The right-to-left implication is trivial. For the left-to-right implication we observe the following:

- $\Rightarrow_{SU}$ is terminating. A suitable lexicographic ordering on the multisets E (with $\bot$ minimal) shows this. Compare in this order:
  (1) the number of variables that occur in E below a function or predicate symbol, or on the right-hand side of an equation, or at least twice;
  (2) the multiset of the sizes (numbers of symbols) of all equations in E;
  (3) the number of non-variable left-hand sides of equations in E.
- A system E that is irreducible w. r. t. $\Rightarrow_{SU}$ is either $\bot$ or a solved form.
- Therefore, reducing any E by SU will end (no matter what reduction strategy we apply) in an irreducible E' having the same unifiers as E, and we can read off the mgu (or non-unifiability) of E from E' (Theorem 3.27, Proposition 3.26).
- σ is idempotent because of the substitution in rule 4. $\mathrm{dom}(\sigma) \cup \mathrm{codom}(\sigma) \subseteq \mathrm{var}(E)$, as no new variables are generated.

□

Rule-Based Polynomial Unification

Problem: using $\Rightarrow_{SU}$, an *exponential growth* of terms is possible.

The following unification algorithm avoids this problem, at least if the final solved form is represented as a DAG.

$$
\begin{array}{rcl}
t \doteq t, E & \Rightarrow_{PU} & E \\
f(s_1, \ldots, s_n) \doteq f(t_1, \ldots, t_n), E & \Rightarrow_{PU} & s_1 \doteq t_1, \ldots, s_n \doteq t_n, E \\
f(\ldots) \doteq g(\ldots), E & \Rightarrow_{PU} & \bot \\
 & & \text{if } f \neq g \\
x \doteq y, E & \Rightarrow_{PU} & x \doteq y, E\{x \mapsto y\} \\
 & & \text{if } x \in \mathrm{var}(E), x \neq y \\
x_1 \doteq t_1, \ldots, x_n \doteq t_n, E & \Rightarrow_{PU} & \bot \\
 & & \text{if there are positions } p_i \text{ with} \\
 & & t_i|_{p_i} = x_{i+1}, t_n|_{p_n} = x_1 \\
 & & \text{and some } p_i \neq \varepsilon \\
x \doteq t, E & \Rightarrow_{PU} & \bot \\
 & & \text{if } x \neq t, x \in \mathrm{var}(t) \\
t \doteq x, E & \Rightarrow_{PU} & x \doteq t, E \\
 & & \text{if } t \notin X \\
x \doteq t, x \doteq s, E & \Rightarrow_{PU} & x \doteq t, t \doteq s, E \\
 & & \text{if } t, s \notin X \text{ and } |t| \leq |s|
\end{array}
$$

Properties of PU

Theorem 3.29

1. *If $E \Rightarrow_{PU} E'$ then σ is a unifier of E iff σ is a unifier of E'*
2. *If $E \Rightarrow^*_{PU} \bot$ then E is not unifiable.*
3. *If $E \Rightarrow^*_{PU} E'$ with E' in solved form, then $\sigma_{E'}$ is an mgu of E.*

Note: The solved form of $\Rightarrow_{PU}$ is different from the solved form obtained from $\Rightarrow_{SU}$. In order to obtain the unifier $\sigma_{E'}$, we have to sort the list of equality problems $x_i \doteq t_i$ in such a way that x_i does not occur in t_j for $j < i$, and then we have to compose the substitutions $\{x_1 \mapsto t_1\} \circ \cdots \circ \{x_k \mapsto t_k\}$.

Resolution for General Clauses

We obtain the resolution inference rules for non-ground clauses from the inference rules for ground clauses by replacing equality by unifiabilty:

General resolution *Res*:

$$\frac{D \vee B \qquad C \vee \neg A}{(D \vee C)\sigma} \quad \text{if } \sigma = \mathrm{mgu}(A, B) \qquad \text{[resolution]}$$

$$\frac{C \vee A \vee B}{(C \vee A)\sigma} \quad \text{if } \sigma = \mathrm{mgu}(A, B) \quad \text{[factorization]}$$

For inferences with more than one premise, we assume that the variables in the premises are (bijectively) renamed such that they become different to any variable in the other premises. We do not formalize this. Which names one uses for variables is otherwise irrelevant.

Lifting Lemma

Lemma 3.30 *Let C and D be variable-disjoint clauses. If*

$$\begin{array}{ccc} D & & C \\ \downarrow \sigma & & \downarrow \rho \\ \end{array}$$

$$\frac{D\sigma \qquad C\rho}{C'} \qquad \textit{[ground resolution]}$$

then there exists a substitution τ such that

$$\frac{D \qquad C}{C''} \qquad \textit{[general resolution]}$$

$$\downarrow \tau$$

$$C' = C''\tau$$

An analogous lifting lemma holds for factorization.

Saturation of Sets of General Clauses

Corollary 3.31 *Let N be a set of general clauses saturated under Res, i. e., $Res(N) \subseteq N$. Then also $G_\Sigma(N)$ is saturated, that is,*

$$Res(G_\Sigma(N)) \subseteq G_\Sigma(N).$$

Proof. W.l.o.g. we may assume that clauses in N are pairwise variable-disjoint. (Otherwise make them disjoint, and this renaming process changes neither $Res(N)$ nor $G_\Sigma(N)$.)

Let $C' \in Res(G_\Sigma(N))$. Then either (i) there exist resolvable ground instances $D\sigma$ and $C\rho$ of N with resolvent C', or else (ii) C' is a factor of a ground instance $C\sigma$ of C.

Case (i): By the Lifting Lemma, D and C are resolvable with a resolvent C'' with $C''\tau = C'$, for a suitable substitution τ. As $C'' \in N$ by assumption, we obtain that $C' \in G_\Sigma(N)$.

Case (ii): Similar. $\square$

Soundness for General Clauses

Proposition 3.32 *The general resolution calculus is sound.*

Proof. We have to show that, if $\sigma = \mathrm{mgu}(A, B)$ then $\{\forall\vec{x}\ (D \vee B), \forall\vec{y}\ (C \vee \neg A)\} \models \forall\vec{z}\ (D \vee C)\sigma$ and $\{\forall\vec{x}\ (C \vee A \vee B)\} \models \forall\vec{z}\ (C \vee A)\sigma$.

Let $\mathcal{A}$ be a model of $\forall\vec{x}\ (D \vee B)$ and $\forall\vec{y}\ (C \vee \neg A)$. By Lemma 3.23, $\mathcal{A}$ is also a model of $\forall\vec{z}\ (D \vee B)\sigma$ and $\forall\vec{z}\ (C \vee \neg A)\sigma$ and by Lemma 3.22, $\mathcal{A}$ is also a model of $(D \vee B)\sigma$ and $(C \vee \neg A)\sigma$. Let β be an assignment. If $\mathcal{A}(\beta)(B\sigma) = 0$, then $\mathcal{A}(\beta)(D\sigma) = 1$. Otherwise $\mathcal{A}(\beta)(B\sigma) = \mathcal{A}(\beta)(A\sigma) = 1$, hence $\mathcal{A}(\beta)(\neg A\sigma) = 0$ and therefore $\mathcal{A}(\beta)(C\sigma) = 1$. In both cases $\mathcal{A}(\beta)((D \vee C)\sigma) = 1$, so $\mathcal{A} \models (D \vee C)\sigma$ and by Lemma 3.22, $\mathcal{A} \models \forall\vec{z}\ (D \vee C)\sigma$.

The proof for factorization inferences is similar. $\square$