max planck institut
informatik

Universität
des
Saarlandes
FR Informatik

Uwe Waldmann **January 11, 2018**

Tutorials for "Automated Reasoning"
Exercise sheet 11

Exercise 11.1: *(5 P)*
Let $(A, \to)$ be a reduction system such that for every $a, b, c \in A$, whenever $b \leftarrow a \to c$ then $b = c$ or there is a $d \in A$ such that $b \to d \leftarrow c$.

Show that if an element $a \in A$ has a normal form, then there is no infinite reduction sequence starting from a.

Exercise 11.2: *(4 P)*
Is the rewrite system

$$\{\, f(a) \to f(b),\ f(b) \to f(c),\ f(c) \to f(a),\ f(x) \to x \,\}$$

(i) terminating, (ii) normalizing, (iii) locally confluent, (iv) confluent? Give a brief explanation.

Exercise 11.3: *(4 P)*
Compute all critical pairs for each of the following term rewrite systems. (Omit the trivial critical pairs obtained by overlapping a rule with itself at the position ε.)

(a) $\{\, f(g(f(x))) \to x,$
$f(g(x)) \to g(f(x)) \,\}$

(b) $\{\, f(x, x) \to b,$
$f(x, g(x)) \to c \,\}$

(c) $\{\, f(f(x, y), z) \to f(x, f(y, z)),$
$f(x, 1) \to x \,\}$

(d) $\{\, f(f(x, y), z) \to f(x, f(y, z)),$
$f(1, x) \to x \,\}$

Which systems are locally confluent?

Exercise 11.4: *(5 P)*
A friend asks you to proofread her master thesis. On page 15 of the thesis, your friend writes the following:

> **Lemma 5.** Let $\succ$ be a well-founded ordering over a set A, let $\to$ be a binary relation such that $\to \subseteq \succ$. Let s and r be two elements of A, such that r is irreducible with respect to $\to$, and define $A_r^s = \{ t \in A \mid s \succeq t,\, t \to^* r \}$. If for every $t_0, t_1, t_2 \in A$ such that $s \succeq t_0$ and $t_1 \leftarrow t_0 \to t_2 \to^* r$ there exists a $t_3 \in A$ such that $t_1 \to^* t_3 \leftarrow^* t_2$, then for every $t_0 \in A_r^s$ and $t_1' \in A$, $t_0 \to^* t_1'$ implies $t_1' \in A_r^s$.
>
> **Proof.** We use well-founded induction over t_0 with respect to $\succ$. Let $t_0 \in A_r^s$ and $t_1' \in A$ such that $t_0 \to^* t_1'$. If this derivation is empty, the result is trivial, so suppose that $t_0 \to t_1 \to^* t_1'$. As $t_0 \in A_r^s$ is reducible, it is different from r, hence there is a non-empty derivation $t_0 \to t_2 \to^* r$. By assumption, there exists a $t_3 \in A$ such that $t_1 \to^* t_3 \leftarrow^* t_2$. Now $t_0 \succ t_2$ and $t_2 \in A_r^s$, hence $t_3 \in A_r^s$ by the induction hypothesis, and thus $t_1 \in A_r^s$. Since $t_0 \succ t_1$, we can use the induction hypothesis once more and obtain $t_1' \in A_r^s$ as required.

(1) Is the "proof" correct (yes/no)?

(2) If the "proof" is not correct:

  (a) Which step is incorrect?

  (b) Does the "theorem" hold? If yes, give a correct proof, otherwise give a counterexample.

Challenge Problem: *(6 Bonus Points)*
Find a signature Σ containing at least one constant symbol, a set E of Σ-equations, and two terms $s, t \in \mathrm{T}_\Sigma(X)$ such that

$$\mathrm{T}_\Sigma(\{x_1\})/E \models \forall \vec{x}(s \approx t),$$

but

$$\mathrm{T}_\Sigma(\{x_1, x_2\})/E \not\models \forall \vec{x}(s \approx t)$$

where $\vec{x}$ consists of all the variables occurring in s and t. (The variables in $\vec{x}$ need not be contained in $\{x_1, x_2\}$.)

Submit your solution in lecture hall E1.3, Room 003 during the lecture on January 18. Please write your name and the time of your tutorial group (Tue 8–10, Wed 8–10, or Wed 14–16) on your solution.

Joint solutions, prepared by up to three persons together, are allowed (but not encouraged). If you prepare your solution jointly, submit it only once and indicate all authors on the sheet.