Assignment 1 *(Multisets)* (8 points)

Prove: If S and S' are finite multisets over a set M, and $S \succ_{\text{mul}} S'$ holds for *every* strict partial ordering $\succ$ over M, then $S' \subset S$ (that is, $S' \subseteq S$ and $S' \neq S$).

Assignment 2 *(Propositional Logic)* (6 + 6 + 6 = 18 points)

Let F, G, H be propositional formulas, let p be a position of H.

Part (a)

Prove or refute: If $H[F]_p$ is valid and $H[G]_p$ is valid, then $H[F \vee G]_p$ is valid.

Part (b)

Prove or refute: If $H[F \wedge G]_p$ is valid, then $H[F]_p$ and $H[G]_p$ are valid.

Part (c)

Prove or refute: If $H[F]_p$ is valid and $\text{pol}(H, p) = -1$, then $H[F \wedge G]_p$ is valid.

Assignment 3 *(CDCL)* (6 + 10 = 16 points)

Let N be the following set of propositional clauses:

$$\begin{array}{ccccccccccccccl}
P & & & \vee & R & & & & & \vee & U & \vee & \neg V & & (1) \\
P & \vee & Q & \vee & \neg R & \vee & S & & & \vee & U & & & & (2) \\
 & & Q & & & \vee & S & & & & & \vee & V & & (3) \\
 & & Q & & & & & & & \vee & \neg U & \vee & \neg V & & (4) \\
 & & Q & & & & & \vee & T & \vee & U & & & & (5) \\
 & & & & & & \neg S & \vee & \neg T & & & & & & (6) \\
 & & & & & & \neg S & & & \vee & \neg U & & & & (7) \\
\neg P & \vee & \neg Q & & & \vee & S & & & & & & & & (8)
\end{array}$$

Part (a)

Use the CDCL procedure to compute a (total) model of N.

Part (b)

Use the CDCL procedure to prove that $N \models P \vee Q$.

For both parts: Use the CDCL inference rules with a reasonable strategy. If you use the *Decide* rule, use the largest undefined positive literal according to the ordering $P > Q > R > S > T > U > V$. If you use the *Backjump* rule, determine a suitable backjump clause using the 1UIP method and use the best possible successor state for that backjump clause.

Assignment 4 *(Propositional Logic, Orderings)* (6 + 6 + 6 = 18 points)

Let Π be a set of propositional variables. We define the relation $\succ$ on propositional formulas over Π by

$$F \succ G \quad \text{if and only if} \quad F \models G \;\text{ and not }\; G \models F$$

Part (a)

Prove: $\succ$ is a strict partial ordering.

Part (b)

Prove: If Π is finite, then $\succ$ is well-founded.

Part (c)

Give an example that demonstrates that $\succ$ is not well-founded if Π is infinite.

Assignment 5 *(First-order Logic)* (10 points)

Let $\Sigma = (\{b/0,\ c/0,\ d/0,\ f/1\}, \{P/1\})$. Does the formula

$$P(b) \land P(c) \land \neg P(d) \land \neg\exists x\, P(f(f(x)))$$

have a Σ-model whose universe has exactly two elements? Give an example of such a model or show that such a model does not exist.

Assignment 6 *(First-order Logic, CNF Transformation)* (10 points)

Let $\Sigma = (\{b/0\}, \{P/1,\ Q/2,\ R/2\})$. Transform the Σ-formula

$$F \;=\; \forall x\, \exists y \Big(\big(P(b) \lor \forall z\, Q(y, z) \big) \rightarrow R(x, y) \Big)$$

into clause normal form using the improved algorithm from Section 3.6. (There are no subformulas in F for which one should introduce a definition.)