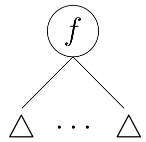
3.7 Herbrand Interpretations

From now on we shall consider FOL without equality. We assume that Ω contains at least one constant symbol.

A *Herbrand interpretation* (over Σ) is a Σ-algebra $\mathcal{A}$ such that

- $U_{\mathcal{A}} = \mathrm{T}_{\Sigma}$ (= the set of ground terms over Σ)
- $f_{\mathcal{A}} : (s_1, \ldots, s_n) \mapsto f(s_1, \ldots, s_n),\ f/n \in \Omega$

$f_{\mathcal{A}}(\triangle, \ldots, \triangle) =$



In other words, *values are fixed* to be ground terms and *functions are fixed* to be the *term constructors*. Only predicate symbols $P/m \in \Pi$ may be freely interpreted as relations $P_{\mathcal{A}} \subseteq \mathrm{T}_{\Sigma}^{m}$.

Proposition 3.10 *Every set of ground atoms I uniquely determines a Herbrand interpretation $\mathcal{A}$ via*

$$(s_1, \ldots, s_n) \in P_{\mathcal{A}} \quad \textit{iff} \quad P(s_1, \ldots, s_n) \in I$$

Thus we shall identify Herbrand interpretations (over Σ) with sets of Σ-ground atoms.

Example: $\Sigma_{Pres} = (\{0/0, s/1, +/2\},\ \{</2, \leq/2\})$

$\mathbb{N}$ as Herbrand interpretation over Σ_{Pres}:

$$\begin{array}{ll} I = \{ & 0 \leq 0,\ 0 \leq s(0),\ 0 \leq s(s(0)),\ \ldots, \\ & 0 + 0 \leq 0,\ 0 + 0 \leq s(0),\ \ldots, \\ & \ldots,\ (s(0) + 0) + s(0) \leq s(0) + (s(0) + s(0)) \\ & \ldots \\ & s(0) + 0 < s(0) + 0 + 0 + s(0) \\ & \ldots\} \end{array}$$

Existence of Herbrand Models

A Herbrand interpretation I is called a *Herbrand model* of F, if $I \models F$.

Theorem 3.11 (Herbrand) *Let N be a set of Σ-clauses.*

$$\begin{array}{lcl} N \textit{ satisfiable} & \Leftrightarrow & N \textit{ has a Herbrand model (over } \Sigma) \\ & \Leftrightarrow & G_\Sigma(N) \textit{ has a Herbrand model (over } \Sigma) \end{array}$$

where $G_\Sigma(N) = \{\, C\sigma \text{ ground clause} \mid C \in N,\ \sigma : X \rightarrow \mathrm{T}_\Sigma \,\}$ *is the set of* ground instances *of* N.

[The proof will be given below in the context of the completeness proof for resolution.]

Example of a G_Σ

For Σ_{Pres} one obtains for

$$C = (x < y) \vee (y \leq s(x))$$

the following ground instances:

$(0 < 0) \vee (0 \leq s(0))$
$(s(0) < 0) \vee (0 \leq s(s(0)))$
$\ldots$
$(s(0) + s(0) < s(0) + 0) \vee (s(0) + 0 \leq s(s(0) + s(0)))$
$\ldots$

3.8 Inference Systems and Proofs

Inference systems Γ (proof calculi) are sets of tuples

$$(F_1, \dots, F_n, F_{n+1}),\ n \geq 0,$$

called *inferences*, and written

$$\frac{\overbrace{F_1\ \dots\ F_n}^{\text{premises}}}{\underbrace{F_{n+1}}_{\text{conclusion}}}\ .$$

Clausal inference system: premises and conclusions are clauses. One also considers inference systems over other data structures.

Inference Systems

Inference systems Γ are shorthands for rewrite systems over sets of formulas. If N is a set of formulas, then

$$\frac{\overbrace{F_1\ \dots\ F_n}^{\text{premises}}}{\underbrace{F_{n+1}}_{\text{conclusion}}}\quad \textit{side condition}$$

is a shorthand for

$$N \cup \{F_1, \dots, F_n\} \quad \Rightarrow_\Gamma \quad N \cup \{F_1, \dots, F_n\} \cup \{F_{n+1}\}$$
$$\text{if } \textit{side condition}$$

Proofs

A *proof* in Γ of a formula F from a a set of formulas N (called *assumptions*) is a sequence $F_1, \dots, F_k$ of formulas where

(i) $F_k = F$,

(ii) for all $1 \leq i \leq k$: $F_i \in N$ or there exists an inference

$$\frac{F_{m_1}\ \dots\ F_{m_n}}{F_i}$$

in Γ, such that $0 \leq m_j < i$, for $1 \leq j \leq n$.

Soundness and Completeness

Provability $\vdash_\Gamma$ of F from N in Γ:

$$N \vdash_\Gamma F \;\text{ if there exists a proof } \Gamma \text{ of } F \text{ from } N.$$

Γ is called *sound*, if

$$\frac{F_1 \;\ldots\; F_n}{F} \in \Gamma \;\text{ implies } F_1, \ldots, F_n \models F$$

Γ is called *complete*, if

$$N \models F \;\text{ implies } N \vdash_\Gamma F$$

Γ is called *refutationally complete*, if

$$N \models \bot \;\text{ implies } N \vdash_\Gamma \bot$$

Proposition 3.12

(i) Let Γ be sound. Then $N \vdash_\Gamma F \Rightarrow N \models F$

(ii) If $N \vdash_\Gamma F$ then there exist finitely many $F_1, \ldots, F_n \in N$ such that $F_1, \ldots, F_n \vdash_\Gamma F$

Proofs as Trees

markings	$\widehat{=}$	formulas			
leaves	$\widehat{=}$	assumptions and axioms			
other nodes	$\widehat{=}$	inferences:	conclusion	$\widehat{=}$	ancestor
			premises	$\widehat{=}$	direct descendants

$$\dfrac{P(f(c)) \qquad \dfrac{\dfrac{\dfrac{P(f(c)) \lor Q(b) \qquad \dfrac{\dfrac{P(f(c)) \lor Q(b) \quad \neg P(f(c)) \lor \neg P(f(c)) \lor Q(b)}{\neg P(f(c)) \lor Q(b) \lor Q(b)}}{\neg P(f(c)) \lor Q(b)}}{Q(b) \lor Q(b)}}{Q(b)} \qquad \neg P(f(c)) \lor \neg Q(b)}{\neg P(f(c))}}{\bot}$$

3.9 Ground (or propositional) Resolution

We observe that propositional clauses and ground clauses are essentially the same, as long as we do not consider equational atoms.

In this section we only deal with ground clauses.

The Resolution Calculus Res

Resolution inference rule:

$$\frac{D \vee A \qquad \neg A \vee C}{D \vee C}$$

Terminology: $D \vee C$: *resolvent*; A: *resolved atom*

(Positive) factorization inference rule:

$$\frac{C \vee A \vee A}{C \vee A}$$

These are *schematic inference rules*; for each substitution of the *schematic variables* C, D, and A, by ground clauses and ground atoms, respectively, we obtain an inference.

We treat "$\vee$" as associative and commutative, hence A and $\neg A$ can occur anywhere in the clauses; moreover, when we write $C \vee A$, etc., this includes unit clauses, that is, $C = \bot$.

Sample Refutation

1.	$\neg P(f(c)) \vee \neg P(f(c)) \vee Q(b)$	(given)
2.	$P(f(c)) \vee Q(b)$	(given)
3.	$\neg P(g(b,c)) \vee \neg Q(b)$	(given)
4.	$P(g(b,c))$	(given)
5.	$\neg P(f(c)) \vee Q(b) \vee Q(b)$	(Res. 2. into 1.)
6.	$\neg P(f(c)) \vee Q(b)$	(Fact. 5.)
7.	$Q(b) \vee Q(b)$	(Res. 2. into 6.)
8.	$Q(b)$	(Fact. 7.)
9.	$\neg P(g(b,c))$	(Res. 8. into 3.)
10.	$\bot$	(Res. 4. into 9.)

Soundness of Resolution

Theorem 3.13 *Propositional resolution is sound.*

Proof. Let $\mathcal{B} \in \Sigma$-Alg. To be shown:

(i) for resolution: $\mathcal{B} \models D \vee A,\ \mathcal{B} \models C \vee \neg A \ \Rightarrow\ \mathcal{B} \models D \vee C$

(ii) for factorization: $\mathcal{B} \models C \vee A \vee A \ \Rightarrow\ \mathcal{B} \models C \vee A$

(i): Assume premises are valid in $\mathcal{B}$. Two cases need to be considered:
If $\mathcal{B} \models A$, then $\mathcal{B} \models C$, hence $\mathcal{B} \models D \vee C$.
Otherwise, $\mathcal{B} \models \neg A$, then $\mathcal{B} \models D$, and again $\mathcal{B} \models D \vee C$.
(ii): even simpler. □

Note: In propositional logic (ground clauses) we have:

1. $\mathcal{B} \models L_1 \vee \ldots \vee L_n$ if and only if there exists i: $\mathcal{B} \models L_i$.
2. $\mathcal{B} \models A$ or $\mathcal{B} \models \neg A$.

This does not hold for formulas with variables!

3.10 Refutational Completeness of Resolution

How to show refutational completeness of propositional resolution:

- We have to show: $N \models \bot \ \Rightarrow\ N \vdash_{Res} \bot$, or equivalently: If $N \not\vdash_{Res} \bot$, then N has a model.
- Idea: Suppose that we have computed sufficiently many inferences (and not derived $\bot$).
- Now order the clauses in N according to some appropriate ordering, inspect the clauses in ascending order, and construct a series of Herbrand interpretations.
- The limit interpretation can be shown to be a model of N.

Clause Orderings

1. We assume that $\succ$ is any fixed ordering on ground atoms that is *total* and *well-founded*. (There exist many such orderings, e. g., the lenght-based ordering on atoms when these are viewed as words over a suitable alphabet.)
2. Extend $\succ$ to an *ordering* $\succ_L$ *on ground literals:*

$$
\begin{array}{lcll}
[\neg]A & \succ_L & [\neg]B & \text{, if } A \succ B \\
\neg A & \succ_L & A &
\end{array}
$$

3. Extend $\succ_L$ to an *ordering* $\succ_C$ *on ground clauses:*
   $\succ_C = (\succ_L)_{\text{mul}}$, the multiset extension of $\succ_L$.

   Notation: $\succ$ also for $\succ_L$ and $\succ_C$.

Example

Suppose $A_5 \succ A_4 \succ A_3 \succ A_2 \succ A_1 \succ A_0$. Then:

$$
\begin{array}{ll}
 & A_5 \vee \neg A_5 \\
\succ & A_3 \vee \neg A_4 \\
\succ & \neg A_1 \vee A_3 \vee A_4 \\
\succ & \neg A_1 \vee A_2 \\
\succ & A_1 \vee A_2 \\
\succ & A_0 \vee A_1
\end{array}
$$

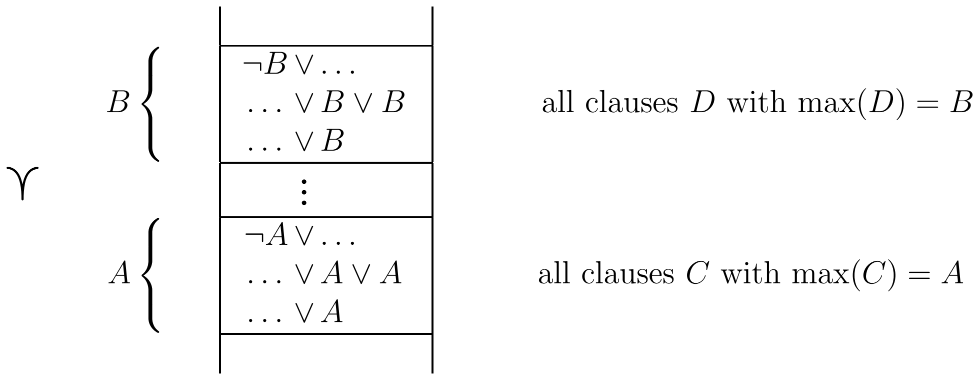
Properties of the Clause Ordering

Proposition 3.14

1. *The orderings on literals and clauses are total and well-founded.*
2. *Let C and D be clauses with $A = \max(C)$, $B = \max(D)$, where $\max(C)$ denotes the maximal atom in C.*
   (i) If $A \succ B$ then $C \succ D$.
   (ii) If $A = B$, A occurs negatively in C but only positively in D, then $C \succ D$.

Stratified Structure of Clause Sets

Let $B \succ A$. Clause sets are then stratified in this form:



Closure of Clause Sets under *Res*

$$
\begin{aligned}
Res(N) &= \{\, C \mid C \text{ is conclusion of an inference in } Res \\
&\qquad\qquad \text{with premises in } N \,\} \\
Res^0(N) &= N \\
Res^{n+1}(N) &= Res(Res^n(N)) \cup Res^n(N), \text{ for } n \geq 0 \\
Res^*(N) &= \textstyle\bigcup_{n \geq 0} Res^n(N)
\end{aligned}
$$

N is called *saturated* (w. r. t. resolution), if $Res(N) \subseteq N$.

Proposition 3.15

(i) $Res^*(N)$ *is saturated.*

(ii) Res *is refutationally complete, iff for each set* N *of ground clauses:*

$$N \models \bot \;\; \textit{iff} \;\; \bot \in Res^*(N)$$

Construction of Interpretations

Given: set N of ground clauses, atom ordering $\succ$.
Wanted: Herbrand interpretation I such that

- "many" clauses from N are valid in I;
- $I \models N$, *if* N *is saturated and* $\bot \notin N$.

Construction according to $\succ$, starting with the minimal clause.

Main Ideas of the Construction

- Clauses are considered in the order given by $\succ$.
- When considering C, one already has a partial interpretation I_C (initially $I_C = \emptyset$) available.
- If C is true in the partial interpretation I_C, nothing is done. ($\Delta_C = \emptyset$).
- If C is false, one would like to change I_C such that C becomes true.
- Changes should, however, be *monotone*. One never deletes anything from I_C and the truth value of clauses smaller than C should be maintained the way it was in I_C.
- Hence, one chooses $\Delta_C = \{A\}$ if, and only if, C is false in I_C, if A occurs positively in C (*adding* A *will make* C *become true*) and if this occurrence in C is strictly maximal in the ordering on literals (*changing the truth value of* A *has no effect on smaller clauses*).

Construction of Candidate Interpretations

Let $N, \succ$ be given. We define sets I_C and Δ_C for all ground clauses C over the given signature inductively over $\succ$:

$$I_C := \bigcup_{C \succ D} \Delta_D$$

$$\Delta_C := \begin{cases} \{A\}, & \text{if } C \in N,\ C = C' \vee A,\ A \succ C',\ I_C \not\models C \\ \emptyset, & \text{otherwise} \end{cases}$$

We say that C *produces* A, if $\Delta_C = \{A\}$.

The *candidate interpretation* for N (w. r. t. $\succ$) is given as $I_N^{\succ} := \bigcup_C \Delta_C$. (We also simply write I_N or I for $I_N^{\succ}$ if $\succ$ is either irrelevant or known from the context.)

Example

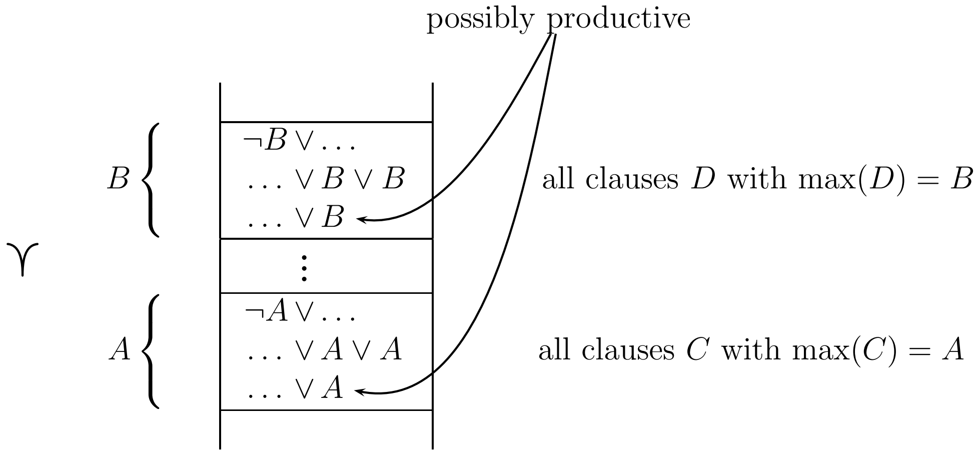
Let $A_5 \succ A_4 \succ A_3 \succ A_2 \succ A_1 \succ A_0$ (max. literals in red)

	clauses C	I_C	Δ_C	Remarks
7	$\neg A_1 \vee A_5$	$\{A_1, A_2, A_4\}$	$\{A_5\}$	
6	$\neg A_1 \vee A_3 \vee \neg A_4$	$\{A_1, A_2, A_4\}$	$\emptyset$	A_3 not maximal; *min. counter-ex.*
5	$A_0 \vee \neg A_1 \vee A_3 \vee A_4$	$\{A_1, A_2\}$	$\{A_4\}$	A_4 maximal
4	$\neg A_1 \vee A_2$	$\{A_1\}$	$\{A_2\}$	A_2 maximal
3	$A_1 \vee A_2$	$\{A_1\}$	$\emptyset$	true in I_C
2	$A_0 \vee A_1$	$\emptyset$	$\{A_1\}$	A_1 maximal
1	$\neg A_0$	$\emptyset$	$\emptyset$	true in I_C

$I = \{A_1, A_2, A_4, A_5\}$ is not a model of the clause set
$\Rightarrow$ there exists a *counterexample*.

Structure of $N, \succ$

Let $B \succ A$; producing a new atom does not affect smaller clauses.



Some Properties of the Construction

Proposition 3.16

(i) *If $C = \neg A \vee C'$, then no $D \succeq C$ produces A.*

(ii) *If C is productive, then $I_C \cup \Delta_C \models C$.*

(iii) *Let $D' \succeq D \succeq C$. Then*

$$I_D \cup \Delta_D \models C \;\; \textit{implies} \;\; I_{D'} \cup \Delta_{D'} \models C \;\; \textit{and} \;\; I_N \models C.$$

If, in addition, $C \in N$ or $\max(D) \succ \max(C)$, then

$$I_D \cup \Delta_D \not\models C \;\; \textit{implies} \;\; I_{D'} \cup \Delta_{D'} \not\models C \;\; \textit{and} \;\; I_N \not\models C.$$

(iv) *Let $D' \succeq D \succ C$. Then*

$$I_D \models C \;\; \textit{implies} \;\; I_{D'} \models C \;\; \textit{and} \;\; I_N \models C.$$

If, in addition, $C \in N$ or $\max(D) \succ \max(C)$, then

$$I_D \not\models C \;\; \textit{implies} \;\; I_{D'} \not\models C \;\; \textit{and} \;\; I_N \not\models C.$$

(v) *If $D = D' \vee A$ produces A, then $I_C \not\models D'$ for every $C \succ D$ and $I_N \not\models D'$.*

(vi) *If every clause $C \in N$ is productive or $I_C \models C$, then $I_N \models N$.*

Resolution Reduces Counterexamples

$$\frac{A_0 \vee \neg A_1 \vee A_3 \vee A_4 \qquad \neg A_1 \vee A_3 \vee \neg A_4}{A_0 \vee \neg A_1 \vee \neg A_1 \vee A_3 \vee A_3}$$

Construction of I for the extended clause set:

clauses C	I_C	Δ_C	Remarks
$\neg A_1 \vee A_5$	$\{A_1, A_2, A_4\}$	$\{A_5\}$	
$\neg A_1 \vee A_3 \vee \neg A_4$	$\{A_1, A_2, A_4\}$	$\emptyset$	counterexample
$A_0 \vee \neg A_1 \vee A_3 \vee A_4$	$\{A_1, A_2\}$	$\{A_4\}$	
$A_0 \vee \neg A_1 \vee \neg A_1 \vee A_3 \vee A_3$	$\{A_1, A_2\}$	$\emptyset$	A_3 occurs twice *minimal counter-ex.*
$\neg A_1 \vee A_2$	$\{A_1\}$	$\{A_2\}$	
$A_1 \vee A_2$	$\{A_1\}$	$\emptyset$	
$A_0 \vee A_1$	$\emptyset$	$\{A_1\}$	
$\neg A_0$	$\emptyset$	$\emptyset$	

The same I, but smaller counterexample, hence some progress was made.

Factorization Reduces Counterexamples

$$\frac{A_0 \vee \neg A_1 \vee \neg A_1 \vee A_3 \vee A_3}{A_0 \vee \neg A_1 \vee \neg A_1 \vee A_3}$$

Construction of I for the extended clause set:

clauses C	I_C	Δ_C	Remarks
$\neg A_1 \vee A_5$	$\{A_1, A_2, A_3\}$	$\{A_5\}$	
$\neg A_1 \vee A_3 \vee \neg A_4$	$\{A_1, A_2, A_3\}$	$\emptyset$	true in I_C
$A_0 \vee \neg A_1 \vee A_3 \vee A_4$	$\{A_1, A_2, A_3\}$	$\emptyset$	
$A_0 \vee \neg A_1 \vee \neg A_1 \vee A_3 \vee A_3$	$\{A_1, A_2, A_3\}$	$\emptyset$	true in I_C
$A_0 \vee \neg A_1 \vee \neg A_1 \vee A_3$	$\{A_1, A_2\}$	$\{A_3\}$	
$\neg A_1 \vee A_2$	$\{A_1\}$	$\{A_2\}$	
$A_1 \vee A_2$	$\{A_1\}$	$\emptyset$	
$A_0 \vee A_1$	$\emptyset$	$\{A_1\}$	
$\neg A_0$	$\emptyset$	$\emptyset$	

The resulting $I = \{A_1, A_2, A_3, A_5\}$ is a model of the clause set.

Model Existence Theorem

Theorem 3.17 (Bachmair & Ganzinger 1990) *Let $\succ$ be a clause ordering, let N be saturated w. r. t. Res, and suppose that $\bot \notin N$. Then $I_N^{\succ} \models N$.*

Proof. Suppose $\bot \notin N$, but $I_N^{\succ} \not\models N$. Let $C \in N$ minimal (w. r. t. $\succ$) such that C is neither productive nor $I_C \models C$. As $C \neq \bot$ there exists a maximal literal in C. There are two possible reasons why C is not productive:

Case 1: The maximal literal $\neg A$ is negative, i. e., $C = \neg A \vee C'$. Then $I_C \models A$ and $I_C \not\models C'$. So some $D = D' \vee A \in N$ with $C \succ D$ produces A, and $I_C \not\models D'$. The inference

$$\frac{D' \vee A \qquad \neg A \vee C'}{D' \vee C'}$$

yields a clause $D' \vee C' \in N$ that is smaller than C. As $I_C \not\models D' \vee C'$, we know that $D' \vee C'$ is neither productive nor $I_{D' \vee C'} \models D' \vee C'$. This contradicts the minimality of C.

Case 2: The maximal literal A is positive, but not strictly maximal, i. e., $C = C' \vee A \vee A$. Then there is an inference

$$\frac{C' \vee A \vee A}{C' \vee A}$$

that yields a smaller clause $C' \vee A \in N$. As $I_C \not\models C' \vee A$, this clause is neither productive nor $I_{C' \vee A} \models C' \vee A$. Since $C \succ C' \vee A$, this contradicts the minimality of C. □

Corollary 3.18 *Let N be saturated w. r. t. Res. Then $N \models \bot$ if and only if $\bot \in N$.*

Compactness of Propositional Logic

Lemma 3.19 *Let N be a set of propositional (or first-order ground) clauses. Then N is unsatisfiable, if and only if some finite subset $N' \subseteq N$ is unsatisfiable.*

Proof. The "if" part is trivial. For the "only if" part, assume that N be unsatisfiable. Consequently, $Res^*(N)$ unsatisfiable as well. By refutational completeness of resolution, $\bot \in Res^*(N)$. So there exists an $n \geq 0$ such that $\bot \in Res^n(N)$, which means that $\bot$ has a finite resolution proof P. Now choose N' as the set of assumptions in P. □

Theorem 3.20 (Compactness for Propositional Formulas) *Let S be a set of propositional (or first-order ground) formulas. Then S is unsatisfiable, if and only if some finite subset $S' \subseteq S$ is unsatisfiable.*

Proof. If "if" part is again trivial. For the "only if" part, assume that S be unsatisfiable. Transform S into an equivalent set N of clauses. By the previous lemma, N has a finite unsatisfiable subset N'. Now choose for every clause C in N' one formula F of S such that C is contained in the CNF of F. Let S' be the set of these formulas. □