3.13 A Resolution Prover

So far: static view on completeness of resolution:

Saturated sets are inconsistent if and only if they contain $\bot$.

We will now consider a dynamic view:

How can we get saturated sets in practice?

The theorems 3.42 and 3.43 are the basis for the completeness proof of our prover RP.

Rules for Simplifications and Deletion

We want to employ the following rules for simplification of prover states N (there are more possibilities):

- *Deletion of tautologies*

$$N \uplus \{C \vee A \vee \neg A\} \;\Rightarrow\; N$$

- *Deletion of subsumed clauses*

$$N \uplus \{C, D\} \;\Rightarrow\; N \cup \{C\}$$

  if $C\sigma \subseteq D$ (C *subsumes* D).

- *Reduction* (also called *subsumption resolution*)

$$N \uplus \{D \vee L,\;\; C \vee D\sigma \vee \overline{L}\sigma\} \;\Rightarrow\; N \cup \{D \vee L, C \vee D\sigma\}$$

Resolution Prover RP

3 clause sets:
N(ew) containing new inferred clauses;
U(sable) containing reduced new inferred clauses;
clauses get into *W(orked) O(ff)* once their inferences have been computed.

Strategy:
Inferences will only be computed when there are no possibilities for simplification.

Transition Rules for RP (I)

Tautology deletion

$(N \uplus \{C\}\,;\, U\,;\; WO) \quad \Rightarrow_{RP} \quad (N\,;\, U\,;\; WO)$

if C is a tautology

Forward subsumption

$(N \uplus \{C\}\,;\, U\,;\; WO) \quad \Rightarrow_{RP} \quad (N\,;\, U\,;\; WO)$

if some $D \in U \cup WO$ subsumes C $(D\sigma \subseteq C)$

Backward subsumption

$(N \uplus \{C\}\,;\, U \uplus \{D\}\,;\; WO) \quad \Rightarrow_{RP} \quad (N \cup \{C\}\,;\, U\,;\; WO)$

$(N \uplus \{C\}\,;\, U\,;\; WO \uplus \{D\}) \quad \Rightarrow_{RP} \quad (N \cup \{C\}\,;\, U\,;\; WO)$

if C strictly subsumes D $(C\sigma \subset D)$

Transition Rules for RP (II)

Forward subsumption resolution

$(N \uplus \{C \vee L\}\,;\, U\,;\; WO) \quad \Rightarrow_{RP} \quad (N \cup \{C\}\,;\, U\,;\; WO)$

if $D \vee L' \in U \cup WO$ such that $\overline{L} = L'\sigma$ and $D\sigma \subseteq C$

Backward subsumption resolution

$(N\,;\, U \uplus \{C \vee L\}\,;\; WO) \quad \Rightarrow_{RP} \quad (N\,;\, U \cup \{C\}\,;\; WO)$

$(N\,;\, U\,;\; WO \uplus \{C \vee L\}) \quad \Rightarrow_{RP} \quad (N\,;\, U \cup \{C\}\,;\; WO)$

if $D \vee L' \in N$ such that $\overline{L} = L'\sigma$ and $D\sigma \subseteq C$

Transition Rules for RP (III)

Clause processing

$(N \uplus \{C\}\,;\, U\,;\; WO) \quad \Rightarrow_{RP} \quad (N\,;\, U \cup \{C\}\,;\; WO)$

Inference computation

$(\emptyset\,;\, U \uplus \{C\}\,;\; WO) \quad \Rightarrow_{RP} \quad (N\,;\, U\,;\; WO \cup \{C\})$

where N is the set of conclusions of $Res^{\succ}_{\text{sel}}$-inferences from clauses in $WO \cup \{C\}$, where C is one of the premises

Soundness and Completeness

Theorem 3.44

$$N \models \bot \quad \Leftrightarrow \quad (N\,;\emptyset\,;\emptyset) \quad \Rightarrow^{*}_{RP} \quad (N' \cup \{\bot\}\,;\,\_\,;\,\_)$$

Proof in L. Bachmair, H. Ganzinger: Resolution Theorem Proving (appeared in the Handbook of Automated Reasoning, 2001)

Fairness

Problem:

If N is inconsistent, then $(N\,;\emptyset\,;\emptyset) \Rightarrow^{*}_{RP} (N' \cup \{\bot\}\,;\,\_\,;\,\_)$.

Does this imply that *every* derivation starting from an inconsistent set N eventually produces $\bot$?

No: a clause could be kept in U without ever being used for an inference.

We need in addition a *fairness condition:*

If an inference is possible forever (that is, none of its premises is ever deleted), then it must be computed eventually.

One possible way to guarantee fairness: Implement U as a queue (there are other techniques to guarantee fairness).

With this additional requirement, we get a stronger result: If N is inconsistent, then every *fair* derivation will eventually produce $\bot$.

3.14 Hyperresolution

There are *many* variants of resolution.

One well-known example is hyperresolution (Robinson 1965):

Assume that several negative literals are selected in a clause C. If we perform an inference with C, then one of the selected literals is eliminated.

Suppose that the remaining selected literals of C are again selected in the conclusion. Then we must eliminate the remaining selected literals one by one by further resolution steps.

Hyperresolution replaces these successive steps by a single inference. As for $Res^{\succ}_{\text{sel}}$, the calculus is parameterized by an atom ordering $\succ$ and a selection function sel.

$$\frac{D_1 \vee B_1 \quad \ldots \quad D_n \vee B_n \qquad C \vee \neg A_1 \vee \ldots \vee \neg A_n}{(D_1 \vee \ldots \vee D_n \vee C)\sigma}$$

with $\sigma = \text{mgu}(A_1 \doteq B_1, \ldots, A_n \doteq B_n)$, if

(i) $B_i\sigma$ strictly maximal in $D_i\sigma$, $1 \leq i \leq n$;

(ii) nothing is selected in D_i;

(iii) the indicated occurrences of the $\neg A_i$ are exactly the ones selected by sel, or else nothing is selected in the right premise and $n = 1$ and $\neg A_1\sigma$ is maximal in $C\sigma$.

Similarly to resolution, hyperresolution has to be complemented by a factorization inference.

As we have seen, hyperresolution can be simulated by iterated binary resolution.

However this yields intermediate clauses which HR might not derive, and many of them might not be extendable into a full HR inference.

3.15 Summary: Resolution Theorem Proving

- Resolution is a machine calculus.
- Subtle interleaving of enumerating instances and proving inconsistency through the use of unification.
- Parameters: atom ordering $\succ$ and selection function sel. On the non-ground level, ordering constraints can (only) be solved approximatively.
- Completeness proof by constructing candidate interpretations from productive clauses $C \vee A$, $A \succ C$; inferences with those reduce counterexamples.
- *Local* restrictions of inferences via $\succ$ and sel
  $\Rightarrow$ fewer proof variants.
- *Global* restrictions of the search space via elimination of redundancy
  $\Rightarrow$ computing with "smaller" clause sets;
  $\Rightarrow$ termination on many decidable fragments.
- However: not good enough for dealing with orderings, equality and more specific algebraic theories (lattices, abelian groups, rings, fields)
  $\Rightarrow$ further specialization of inference systems required.