Knuth-Bendix Completion: Inference Rules

Notations:

The formula $s \mathrel{\dot{\approx}} t$ denotes either $s \approx t$ or $t \approx s$.

$\mathrm{CP}(R)$ denotes the set of all critical pairs between rules in R.

Knuth-Bendix Completion: Inference Rules

Orient

$$(E \uplus \{s \mathrel{\dot{\approx}} t\}; R) \quad \Rightarrow_{KBC} \quad (E; R \cup \{s \to t\})$$

if $s \succ t$

Note: There are equations $s \approx t$ that cannot be oriented, i. e., neither $s \succ t$ nor $t \succ s$.

Knuth-Bendix Completion: Inference Rules

Trivial equations cannot be oriented – but we don't need them anyway:

Delete

$$(E \uplus \{s \approx s\}; R) \quad \Rightarrow_{KBC} \quad (E; R)$$

Knuth-Bendix Completion: Inference Rules

Critical pairs between rules in R are turned into additional equations:

Deduce

$$(E; R) \quad \Rightarrow_{KBC} \quad (E \cup \{s \approx t\}; R)$$

if $\langle s, t \rangle \in \mathrm{CP}(R)$

Note: If $\langle s, t \rangle \in \mathrm{CP}(R)$ then $s \ {}_R\!\leftarrow \ u \ \rightarrow_R \ t$ and hence $R \models s \approx t$.

Knuth-Bendix Completion: Inference Rules

The following inference rules are not absolutely necessary, but very useful (e. g., to get rid of joinable critical pairs and to deal with equations that cannot be oriented):

Simplify-Eq

$$(E \uplus \{s \mathrel{\dot{\approx}} t\}; R) \quad \Rightarrow_{KBC} \quad (E \cup \{u \approx t\}; R)$$

if $s \rightarrow_R u$

Knuth-Bendix Completion: Inference Rules

Simplification of the right-hand side of a rule is unproblematic.

R-Simplify-Rule

$$(E; R \uplus \{s \to t\}) \quad \Rightarrow_{KBC} \quad (E; R \cup \{s \to u\})$$

if $t \to_R u$

Simplification of the left-hand side may influence orientability and orientation. Therefore, it yields an *equation*:

L-Simplify-Rule

$$(E; R \uplus \{s \to t\}) \quad \Rightarrow_{KBC} \quad (E \cup \{u \approx t\}; R$$

if $s \to_R u$ using a rule $l \to r \in R$ such that $s \sqsupset l$ (see next slide).

Knuth-Bendix Completion: Inference Rules

For technical reasons, the lhs of $s \to t$ may only be simplified using a rule $l \to r$, if $l \to r$ *cannot* be simplified using $s \to t$, that is, if $s \sqsupset l$, where the encompassment quasi-ordering $\underset{\sim}{\sqsupset}$ is defined by

$$s \underset{\sim}{\sqsupset} l \quad \text{if} \quad s|_p = l\sigma \text{ for some } p \text{ and } \sigma$$

and $\sqsupset \,=\, \underset{\sim}{\sqsupset} \setminus \underset{\sim}{\sqsubset}$ is the strict part of $\underset{\sim}{\sqsupset}$.

Lemma 4.27:
$\sqsupset$ is a well-founded strict partial ordering.

Knuth-Bendix Completion: Inference Rules

Lemma 4.28:

If $(E; R) \Rightarrow_{KBC} (E'; R')$, then $\approx_{E \cup R} = \approx_{E' \cup R'}$.

Lemma 4.29:

If $(E; R) \Rightarrow_{KBC} (E'; R')$ and $\rightarrow_R \subseteq \succ$, then $\rightarrow_{R'} \subseteq \succ$.

Knuth-Bendix Completion: Correctness Proof

If we run the completion procedure on a set E of equations, different things can happen:

(1) We reach a state where no more inference rules are applicable and E is not empty.
$\Rightarrow$ Failure (try again with another ordering?)

(2) We reach a state where E is empty and all critical pairs between the rules in the current R have been checked.

(3) The procedure runs forever.

In order to treat these cases simultaneously, we need some definitions.

Knuth-Bendix Completion: Correctness Proof

A (finite or infinite sequence) $(E_0; R_0) \Rightarrow_{KBC} (E_1; R_1) \Rightarrow_{KBC} (E_2; R_2) \Rightarrow_{KBC} \ldots$ with $R_0 = \emptyset$ is called a run of the completion procedure with input E_0 and $\succ$.

For a run, $E_\infty = \bigcup_{i \geq 0} E_i$ and $R_\infty = \bigcup_{i \geq 0} R_i$.

The sets of persistent equations or rules of the run are $E_* = \bigcup_{i \geq 0} \bigcap_{j \geq i} E_j$ and $R_* = \bigcup_{i \geq 0} \bigcap_{j \geq i} R_j$.

Note: If the run is finite and ends with E_n, R_n, then $E_* = E_n$ and $R_* = R_n$.

Knuth-Bendix Completion: Correctness Proof

A run is called fair, if $CP(R_*) \subseteq E_\infty$ (i. e., if every critical pair between persisting rules is computed at some step of the derivation).

Goal:

Show: If a run is fair and E_* is empty, then R_* is convergent and equivalent to E_0.

In particular: If a run is fair and E_* is empty, then $\approx_{E_0} = \approx_{E_\infty \cup R_\infty} = \leftrightarrow^*_{E_\infty \cup R_\infty} = \downarrow_{R_*}$.

Knuth-Bendix Completion: Correctness Proof

General assumptions from now on:

$$(E_0; R_0) \Rightarrow_{KBC} (E_1; R_1) \Rightarrow_{KBC} (E_2; R_2) \Rightarrow_{KBC} \ldots$$

is a fair run.

R_0 and E_* are empty.

Knuth-Bendix Completion: Correctness Proof

A proof of $s \approx t$ in $E_\infty \cup R_\infty$ is a finite sequence $(s_0, \dots, s_n)$ such that $s = s_0$, $t = s_n$, and for all $i \in \{1, \dots, n\}$:

(1) $s_{i-1} \leftrightarrow_{E_\infty} s_i$, or

(2) $s_{i-1} \rightarrow_{R_\infty} s_i$, or

(3) $s_{i-1} \; {}_{R_\infty}\!\!\leftarrow s_i$.

The pairs (s_{i-1}, s_i) are called proof steps.

A proof is called a rewrite proof in R_*, if there is a $k \in \{0, \dots, n\}$ such that $s_{i-1} \rightarrow_{R_*} s_i$ for $1 \le i \le k$ and $s_{i-1} \; {}_{R_*}\!\!\leftarrow s_i$ for $k + 1 \le i \le n$

Knuth-Bendix Completion: Correctness Proof

Idea (Bachmair, Dershowitz, Hsiang):

Define a well-founded ordering on proofs, such that for every proof that is not a rewrite proof in R_* there is an equivalent smaller proof.

Consequence: For every proof there is an equivalent rewrite proof in R_*.

Knuth-Bendix Completion: Correctness Proof

We associate a cost $c(s_{i-1}, s_i)$ with every proof step as follows:

(1) If $s_{i-1} \leftrightarrow_{E_\infty} s_i$, then $c(s_{i-1}, s_i) = (\{s_{i-1}, s_i\}, -, -)$, where the first component is a multiset of terms and $-$ denotes an arbitrary (irrelevant) term.

(2) If $s_{i-1} \rightarrow_{R_\infty} s_i$ using $l \rightarrow r$, then $c(s_{i-1}, s_i) = (\{s_{i-1}\}, l, s_i)$.

(3) If $s_{i-1} \;{}_{R_\infty}\!\!\leftarrow s_i$ using $l \rightarrow r$, then $c(s_{i-1}, s_i) = (\{s_i\}, l, s_{i-1})$.

Proof steps are compared using the lexicographic combination of the multiset extension of the reduction ordering $\succ$, the encompassment ordering $\sqsupset$, and the reduction ordering $\succ$.

Knuth-Bendix Completion: Correctness Proof

The cost $c(P)$ of a proof P is the multiset of the costs of its proof steps.

The proof ordering $\succ_C$ compares the costs of proofs using the multiset extension of the proof step ordering.

Lemma 4.30:
$\succ_C$ is a well-founded ordering.

Knuth-Bendix Completion: Correctness Proof

Lemma 4.31:
Let P be a proof in $E_\infty \cup R_\infty$. If P is not a rewrite proof in R_*, then there exists an equivalent proof P' in $E_\infty \cup R_\infty$ such that $P \succ_C P'$.

Proof:
If P is not a rewrite proof in R_*, then it contains

(a) a proof step that is in E_∞, or
(b) a proof step that is in $R_\infty \setminus R_*$, or
(c) a subproof $s_{i-1} \;{}_{R_*}\!\!\leftarrow s_i \rightarrow_{R_*} s_{i+1}$ (peak).

We show that in all three cases the proof step or subproof can be replaced by a smaller subproof:

Knuth-Bendix Completion: Correctness Proof

Case (a): A proof step using an equation $s \doteq t$ is in E_∞. This equation must be deleted during the run.

If $s \doteq t$ is deleted using *Orient*:

$$\dots s_{i-1} \leftrightarrow_{E_\infty} s_i \dots \qquad \Longrightarrow \qquad \dots s_{i-1} \rightarrow_{R_\infty} s_i \dots$$

If $s \doteq t$ is deleted using *Delete*:

$$\dots s_{i-1} \leftrightarrow_{E_\infty} s_{i-1} \dots \qquad \Longrightarrow \qquad \dots s_{i-1} \dots$$

If $s \doteq t$ is deleted using *Simplify-Eq*:

$$\dots s_{i-1} \leftrightarrow_{E_\infty} s_i \dots \qquad \Longrightarrow \qquad \dots s_{i-1} \rightarrow_{R_\infty} s' \leftrightarrow_{E_\infty} s_i \dots$$

Knuth-Bendix Completion: Correctness Proof

Case (b): A proof step using a rule $s \to t$ is in $R_\infty \setminus R_*$. This rule must be deleted during the run.

If $s \to t$ is deleted using *R-Simplify-Rule*:

$$\dots s_{i-1} \to_{R_\infty} s_i \dots \quad \Longrightarrow \quad \dots s_{i-1} \to_{R_\infty} s' \; {}_{R_\infty}\!\!\leftarrow s_i \dots$$

If $s \to t$ is deleted using *L-Simplify-Rule*:

$$\dots s_{i-1} \to_{R_\infty} s_i \dots \quad \Longrightarrow \quad \dots s_{i-1} \to_{R_\infty} s' \leftrightarrow_{E_\infty} s_i \dots$$

Knuth-Bendix Completion: Correctness Proof

Case (c): A subproof has the form $s_{i-1} \;{}_{R_*}\!\!\leftarrow s_i \rightarrow_{R_*} s_{i+1}$.

If there is no overlap or a non-critical overlap:

$$\dots s_{i-1} \;{}_{R_*}\!\!\leftarrow s_i \rightarrow_{R_*} s_{i+1} \dots \implies \dots s_{i-1} \rightarrow^*_{R_*} s' \;{}_{R_*}^{\;*}\!\!\leftarrow s_{i+1} \dots$$

If there is a critical pair that has been added using *Deduce*:

$$\dots s_{i-1} \;{}_{R_*}\!\!\leftarrow s_i \rightarrow_{R_*} s_{i+1} \dots \implies \qquad \dots s_{i-1} \leftrightarrow_{E_\infty} s_{i+1} \dots$$

In all cases, checking that the replacement subproof is smaller than the replaced subproof is routine. □

Knuth-Bendix Completion: Correctness Proof

Theorem 4.32:
Let $(E_0; R_0) \Rightarrow_{KBC} (E_1; R_1) \Rightarrow_{KBC} (E_2; R_2) \Rightarrow_{KBC} \ldots$ be a fair run and let R_0 and E_* be empty. Then

(1) every proof in $E_\infty \cup R_\infty$ is equivalent to a rewrite proof in R_*,

(2) R_* is equivalent to E_0, and

(3) R_* is convergent.

Knuth-Bendix Completion: Correctness Proof

Proof:

(1) By well-founded induction on $\succ_C$ using the previous lemma.

(2) Clearly $\approx_{E_\infty \cup R_\infty} \; = \; \approx_{E_0}$. Since $R_* \subseteq R_\infty$, we get $\approx_{R_*} \subseteq \approx_{E_\infty \cup R_\infty}$. On the other hand, by (1), $\approx_{E_\infty \cup R_\infty} \subseteq \approx_{R_*}$.

(3) Since $\rightarrow_{R_*} \subseteq \succ$, R_* is terminating. By (1), R_* is confluent.

$\square$

4.6 Unfailing Completion

Classical completion:

Try to transform a set E of equations into an equivalent convergent TRS.

Fail, if an equation can neither be oriented nor deleted.

Unfailing completion (Bachmair, Dershowitz and Plaisted):

If an equation cannot be oriented, we can still use *orientable instances* for rewriting.

Note: If $\succ$ is total on ground terms, then every *ground instance* of an equation is trivial or can be oriented.

Goal: Derive a *ground convergent* set of equations.

Unfailing Completion

Let E be a set of equations, let $\succ$ be a reduction ordering.

We define the relation $\rightarrow_{E^\succ}$ by

$$s \rightarrow_{E^\succ} t \quad \text{iff} \quad \begin{array}{l} \text{there exist } (u \approx v) \in E \text{ or } (v \approx u) \in E, \\ p \in \mathrm{pos}(s), \text{ and } \sigma : X \rightarrow \mathrm{T}_\Sigma(X), \\ \text{such that } s|_p = u\sigma \text{ and } t = s[v\sigma]_p \\ \text{and } u\sigma \succ v\sigma. \end{array}$$

Note: $\rightarrow_{E^\succ}$ is terminating by construction.

Unfailing Completion

From now on let $\succ$ be a reduction ordering that is total on ground terms.

E is called ground convergent w. r. t. $\succ$, if for all ground terms s and t with $s \leftrightarrow_E^* t$ there exists a ground term v such that $s \rightarrow_{E^\succ}^* v \; {}_{E^\succ}^{*}\!\!\leftarrow t$.

(Analogously for $E \cup R$.)

Unfailing Completion

As for standard completion, we establish ground convergence by computing critical pairs.

However, the ordering $\succ$ is not total on non-ground terms.

Since $s\theta \succ t\theta$ implies $s \not\preceq t$, we approximate $\succ$ on ground terms by $\not\preceq$ on arbitrary terms.

Unfailing Completion

Let $u_i \mathrel{\dot{\approx}} v_i$ $(i = 1, 2)$ be equations in E whose variables have been renamed such that $\mathrm{vars}(u_1 \mathrel{\dot{\approx}} v_1) \cap \mathrm{vars}(u_2 \mathrel{\dot{\approx}} v_2) = \emptyset$. Let $p \in \mathrm{pos}(u_1)$ be a position such that $u_1|_p$ is not a variable, σ is an mgu of $u_1|_p$ and u_2, and $u_i\sigma \not\preceq v_i\sigma$ $(i = 1, 2)$. Then $\langle v_1\sigma, (u_1\sigma)[v_2\sigma]_p \rangle$ is called a semi-critical pair of E with respect to $\succ$.

The set of all semi-critical pairs of E is denoted by $\mathrm{SP}_\succ(E)$.

Semi-critical pairs of $E \cup R$ are defined analogously. If $\rightarrow_R \subseteq \succ$, then $\mathrm{CP}(R)$ and $\mathrm{SP}_\succ(R)$ agree.

Unfailing Completion

Note: In contrast to critical pairs, it may be necessary to consider overlaps of a rule with itself at the top.

For instance, if $E = \{f(x) \approx g(y)\}$, then $\langle g(y), g(y') \rangle$ is a non-trivial semi-critical pair.

Unfailing Completion

The *Deduce* rule takes now the following form:

Deduce

$$(E; R) \quad \Rightarrow_{UKBC} \quad (E \cup \{s \approx t\}; R)$$

if $\langle s, t \rangle \in \mathrm{SP}_{\succ}(E \cup R)$

The other rules are inherited from $\Rightarrow_{KBC}$. The fairness criterion for runs is replaced by

$$\mathrm{SP}_{\succ}(E_* \cup R_*) \subseteq E_\infty$$

(i. e., if every semi-critical pair between persisting rules or equations is computed at some step of the derivation).

Unfailing Completion

Analogously to Thm. 4.32 we obtain now the following theorem:

Theorem 4.33:
Let $(E_0; R_0) \Rightarrow_{UKBC} (E_1; R_1) \Rightarrow_{UKBC} (E_2; R_2) \Rightarrow_{UKBC} \ldots$ be a fair run; let $R_0 = \emptyset$. Then

(1) $E_* \cup R_*$ is equivalent to E_0, and

(2) $E_* \cup R_*$ is ground convergent.

Unfailing Completion

Moreover one can show that, whenever there exists a *reduced* convergent R such that $\approx_{E_0} = \downarrow_R$ and $\rightarrow_R \in \succ$, then for every fair *and simplifying* run $E_* = \emptyset$ and $R_* = R$ up to variable renaming.

Here R is called reduced, if for every $l \rightarrow r \in R$, both l and r are irreducible w. r. t. $R \setminus \{l \rightarrow r\}$. A run is called simplifying, if R_* is reduced, and for all equations $u \approx v \in E_*$, u and v are incomparable w. r. t. $\succ$ and irreducible w. r. t. R_*.

Unfailing Completion

Unfailing completion is refutationally complete for equational theories:

Theorem 4.34:
Let E be a set of equations, let $\succ$ be a reduction ordering that is total on ground terms. For any two terms s and t, let $\hat{s}$ and $\hat{t}$ be the terms obtained from s and t by replacing all variables by Skolem constants. Let $eq/2$, $true/0$ and $false/0$ be new operator symbols, such that $true$ and $false$ are smaller than all other terms. Let $E_0 = E \cup \{eq(\hat{s}, \hat{t}) \approx true, eq(x, x) \approx false\}$. If $(E_0; \emptyset) \Rightarrow_{UKBC} (E_1; R_1) \Rightarrow_{UKBC} (E_2; R_2) \Rightarrow_{UKBC} \ldots$ be a fair run of unfailing completion, then $s \approx_E t$ iff some $E_i \cup R_i$ contains $true \approx false$.

Unfailing Completion

Outlook:

Combine ordered resolution and unfailing completion to get a calculus for equational clauses:

compute inferences between (strictly) maximal literals as in ordered resolution,

compute overlaps between maximal sides of equations as in unfailing completion

$\Rightarrow$ Superposition calculus.