Rule-Based Naive Standard Unification

$$
\begin{array}{rcl}
t \doteq t, E & \Rightarrow_{SU} & E \\
f(s_1, \ldots, s_n) \doteq f(t_1, \ldots, t_n), E & \Rightarrow_{SU} & s_1 \doteq t_1, \ldots, s_n \doteq t_n, E \\
f(\ldots) \doteq g(\ldots), E & \Rightarrow_{SU} & \perp \\
x \doteq t, E & \Rightarrow_{SU} & x \doteq t, E\{x \mapsto t\} \\
 & & \text{if } x \in var(E), x \notin var(t) \\
x \doteq t, E & \Rightarrow_{SU} & \perp \\
 & & \text{if } x \neq t, x \in var(t) \\
t \doteq x, E & \Rightarrow_{SU} & x \doteq t, E \\
 & & \text{if } t \notin X
\end{array}
$$

SU: Main Properties

If $E = x_1 \doteq u_1, \dots, x_k \doteq u_k$, with x_i pairwise distinct, $x_i \notin var(u_j)$, then E is called an (equational problem in) solved form representing the solution $\sigma_E = \{x_1 \mapsto u_1, \dots, x_k \mapsto u_k\}$.

Proposition 3.22:
If E is a solved form then σ_E is an mgu of E.

SU: Main Properties

Theorem 3.23:

1. If $E \Rightarrow_{SU} E'$ then σ is a unifier of E iff σ is a unifier of E'
2. If $E \Rightarrow^{*}_{SU} \bot$ then E is not unifiable.
3. If $E \Rightarrow^{*}_{SU} E'$ with E' in solved form, then $\sigma_{E'}$ is an mgu of E.

Proof:

(1) We have to show this for each of the rules. Let's treat the case for the 4th rule here. Suppose σ is a unifier of $x \doteq t$, that is, $x\sigma = t\sigma$. Thus, $\sigma \circ \{x \mapsto t\} = \sigma[x \mapsto t\sigma] = \sigma[x \mapsto x\sigma] = \sigma$. Therefore, for any equation $u \doteq v$ in E: $u\sigma = v\sigma$, iff $u\{x \mapsto t\}\sigma = v\{x \mapsto t\}\sigma$. (2) and (3) follow by induction from (1) using Proposition 3.22. $\square$

Main Unification Theorem

Theorem 3.24:
E is unifiable if and only if there is a most general unifier σ of E, such that σ is idempotent and $\mathit{dom}(\sigma) \cup \mathit{codom}(\sigma) \subseteq \mathit{var}(E)$.

Rule-Based Polynomial Unification

Problem: using $\Rightarrow_{SU}$, an *exponential growth* of terms is possible.

The following unification algorithm avoids this problem, at least if the final solved form is represented as a DAG.

Rule-Based Polynomial Unification

$$
\begin{array}{rcl}
t \doteq t, E & \Rightarrow_{PU} & E \\
f(s_1, \ldots, s_n) \doteq f(t_1, \ldots, t_n), E & \Rightarrow_{PU} & s_1 \doteq t_1, \ldots, s_n \doteq t_n, E \\
f(\ldots) \doteq g(\ldots), E & \Rightarrow_{PU} & \bot \\
x \doteq y, E & \Rightarrow_{PU} & x \doteq y, E\{x \mapsto y\} \\
& & \text{if } x \in var(E), x \neq y \\
x_1 \doteq t_1, \ldots, x_n \doteq t_n, E & \Rightarrow_{PU} & \bot \\
& & \text{if there are positions } p_i \text{ with} \\
& & t_i/p_i = x_{i+1}, t_n/p_n = x_1 \\
& & \text{and some } p_i \neq \epsilon
\end{array}
$$

Rule-Based Polynomial Unification

$$x \doteq t, E \quad \Rightarrow_{PU} \quad \bot$$

$$\text{if } x \neq t, x \in var(t)$$

$$t \doteq x, E \quad \Rightarrow_{PU} \quad x \doteq t, E$$

$$\text{if } t \notin X$$

$$x \doteq t, x \doteq s, E \quad \Rightarrow_{PU} \quad x \doteq t, t \doteq s, E$$

$$\text{if } t, s \notin X \text{ and } |t| \leq |s|$$

Properties of PU

Theorem 3.25:

1. If $E \Rightarrow_{PU} E'$ then σ is a unifier of E iff σ is a unifier of E'
2. If $E \Rightarrow^*_{PU} \bot$ then E is not unifiable.
3. If $E \Rightarrow^*_{PU} E'$ with E' in solved form, then $\sigma_{E'}$ is an mgu of E.

Note: The solved form of $\Rightarrow_{PU}$ is different form the solved form obtained from $\Rightarrow_{SU}$. In order to obtain the unifier $\sigma_{E'}$, we have to sort the list of equality problems $x_i \doteq t_i$ in such a way that x_i does not occur in t_j for $j < i$, and then we have to compose the substitutions $\{x_1 \mapsto t_1\} \circ \cdots \circ \{x_k \mapsto t_k\}$.

Lifting Lemma

Lemma 3.26:

Let C and D be variable-disjoint clauses. If

$$\begin{array}{ccc} D & & C \\ \downarrow \sigma & & \downarrow \rho \\ \multicolumn{3}{c}{\dfrac{D\sigma \qquad\qquad C\rho}{C'}} \end{array} \qquad \text{[propositional resolution]}$$

then there exists a substitution τ such that

$$\begin{array}{c} \dfrac{D \qquad\qquad C}{C''} \\ \downarrow \tau \\ C' = C''\tau \end{array} \qquad \text{[general resolution]}$$

Lifting Lemma

An analogous lifting lemma holds for factorization.