2.5 Superposition for $\mathrm{PROP}(\Sigma)$

Superposition for $\mathrm{PROP}(\Sigma)$ is:

- resolution (Robinson 1965) $+$
- ordering restrictions (Bachmair & Ganzinger 1990) $+$
- abstract redundancy critrion (B&G 1990) $+$
- partial model construction (B & G 1990) $+$
- partial-model based inference restriction (Weidenbach)

Resolution for PROP(Σ)

A calculus is a set of inference and reduction rules for a given logic (here PROP(Σ)).

We only consider calculi operating on a set of clauses N. Inference rules *add* new clauses to N whereas reduction rules *remove* clauses from N or *replace* clauses by "simpler" ones.

We are only interested in unsatisfiability, i.e., the considered calculi test whether a clause set N is unsatisfiable. So, in order to check validity of a formula ϕ we check unsatisfiability of the clauses generated from $\neg\phi$.

Resolution for PROP(Σ)

For clauses we switch between the notation as a disjunction, e.g., $P \vee Q \vee P \vee \neg R$, and the notation as a multiset, e.g., $\{P, Q, P, \neg R\}$. This makes no difference as we consider $\vee$ in the context of clauses always modulo AC. Note that $\bot$, the empty disjunction, corresponds to $\emptyset$, the empty multiset.

For literals we write L, possibly with subscript.. If $L = P$ then $\bar{L} = \neg P$ and if $L = \neg P$ then $\bar{L} = P$, so the bar flips the negation of a literal.

Clauses are typically denoted by letters C, D, possibly with subscript.

Resolution for PROP(Σ)

The resolution calculus consists of the inference rules resolution and factoring:

Resolution

$$\mathcal{I}\ \frac{C_1 \lor P \quad C_2 \lor \neg P}{C_1 \lor C_2}$$

Factoring

$$\mathcal{I}\ \frac{C \lor L \lor L}{C \lor L}$$

where C_1, C_2, C always stand for clauses, all inference/reduction rules are applied with respect to AC of $\lor$. Given a clause set N the schema above the inference bar is mapped to N and the resulting clauses below the bar are then *added* to N.

Resolution for PROP(Σ)

and the reduction rules subsumption and tautology deletion:

Subsumption	Tautology Deletion
$\mathcal{R}\dfrac{C_1 \quad C_2}{C_1}$	$\mathcal{R}\dfrac{C \vee P \vee \neg P}{}$

where for subsumption we assume $C_1 \subseteq C_2$. Given a clause set N the schema above the reduction bar is mapped to N and the resulting clauses below the bar *replace* the clauses above the bar in N.

Clauses that can be removed are called redundant.

Resolution for PROP(Σ)

So, if we consider clause sets N as states, $\uplus$ is disjoint union, we get the rules

Resolution $(N \uplus \{C_1 \vee P, C_2 \vee \neg P\}) \Rightarrow (N \cup \{C_1 \vee P, C_2 \vee \neg P\} \cup \{C_1 \vee C_2\})$

Factoring $(N \uplus \{C \vee L \vee L\}) \Rightarrow (N \cup \{C \vee L \vee L\} \cup \{C \vee L\})$

Resolution for PROP(Σ)

Subsumption $(N \uplus \{C_1, C_2\}) \quad \Rightarrow \quad (N \cup \{C_1\})$

provided $C_1 \subseteq C_2$

Tautology Deletion $(N \uplus \{C \vee P \vee \neg P\}) \quad \Rightarrow \quad (N)$

We need more structure than just (N) in order to define a useful rewrite system. We fix this later on.

Resolution for PROP(Σ)

Theorem 2.11:

The resolution calculus is sound and complete:

$$N \text{ is unsatisfiable iff } N \Rightarrow^* \{\bot\}$$

Proof:

Will be a consequence of soundness and completeness of superposition. □

Ordering restrictions

Let $\prec$ be a total ordering on Σ.

We lift $\prec$ to a total ordering on literals by $\prec \subseteq \prec_L$ and $P \prec_L \neg P$ and $\neg P \prec_L Q$ for all $P \prec Q$.

We further lift $\prec_L$ to a total ordering on clauses $\prec_C$ by considering the multiset extension of $\prec_L$ for clauses.

Eventually, we overload $\prec$ with $\prec_L$ and $\prec_C$.

We define $N^{\prec C} = \{D \in N \mid D \prec C\}$.

Ordering restrictions

Eventually we will restrict inferences to maximal literals with respect to $\prec$.

Abstract Redundancy

A clause C is redundant with respect to a clause set N if $N^{\prec C} \models C$.

Tautologies are redundant. Subsumed clauses are redundant if $\subseteq$ is strict.

Remark: Note that for finite N, $N^{\prec C} \models C$ can be decided for PROP(Σ) but is as hard as testing unsatisfiability for a clause set N.

Partial Model Construction

Given a clause set N and an ordering $\prec$ we can construct a (partial) model $N_{\mathcal{I}}$ for N as follows:

$$N_C := \bigcup_{D \prec C} \delta_D$$

$$\delta_D := \begin{cases} \{P\} & \text{if } D = D' \lor P \text{ and } P \text{ maximal and } N_D \not\models D \\ \emptyset & \text{otherwise} \end{cases}$$

$$N_{\mathcal{I}} := \bigcup_{C \in N} \delta_C$$

Superposition

The superposition calculus consists of the inference rules superposition left and factoring:

Superposition Left $(N \uplus \{C_1 \vee P, C_2 \vee \neg P\}) \quad \Rightarrow \quad (N \cup \{C_1 \vee P, C_2 \vee \neg P\} \cup \{C_1 \vee C_2\})$

where P is strictly maximal in $C_1 \vee P$ and $\neg P$ is maximal in $C_2 \vee \neg P$

Factoring $(N \uplus \{C \vee P \vee P\}) \quad \Rightarrow \quad (N \cup \{C \vee P \vee P\} \cup \{C \vee P\})$

where P is maximal in $C \vee P \vee P$

Superposition

examples for specific redundancy rules are

Subsumption $(N \uplus \{C_1, C_2\}) \quad \Rightarrow \quad (N \cup \{C_1\})$

provided $C_1 \subset C_2$

Tautology Deletion $(N \uplus \{C \lor P \lor \neg P\}) \quad \Rightarrow \quad (N)$

Subsumption Resolution $(N \uplus \{C_1 \lor L, C_2 \lor \bar{L}\}) \quad \Rightarrow \quad (N \cup \{C_1 \lor L, C_2\})$

where $C_1 \subseteq C_2$

Superposition

Theorem 2.12:

If from a clause set N all possible superposition inferences are redundant and $\bot \notin N$ then N is satisfiable and $N_{\mathcal{I}} \models N$.