## Properties of PU

**Theorem 3.25**

1. *If $E \Rightarrow_{PU} E'$ then $\sigma$ is a unifier of $E$ iff $\sigma$ is a unifier of $E'$*
2. *If $E \Rightarrow^{*}_{PU} \bot$ then $E$ is not unifiable.*
3. *If $E \Rightarrow^{*}_{PU} E'$ with $E'$ in solved form, then $\sigma_{E'}$ is an mgu of $E$.*

Note: The solved form of $\Rightarrow_{PU}$ is different form the solved form obtained from $\Rightarrow_{SU}$. In order to obtain the unifier $\sigma_{E'}$, we have to sort the list of equality problems $x_i \doteq t_i$ in such a way that $x_i$ does not occur in $t_j$ for $j < i$, and then we have to compose the substitutions $\{x_1 \mapsto t_1\} \circ \cdots \circ \{x_k \mapsto t_k\}$.

## Lifting Lemma

**Lemma 3.26** *Let $C$ and $D$ be variable-disjoint clauses. If*

$$\begin{array}{c} D \qquad\qquad C \\ \big\downarrow \sigma \qquad\quad \big\downarrow \rho \\ \dfrac{D\sigma \qquad\quad C\rho}{C'} \end{array} \qquad \textit{[propositional resolution]}$$

*then there exists a substitution $\tau$ such that*

$$\begin{array}{c} \dfrac{D \qquad\quad C}{C''} \\ \big\downarrow \tau \\ C' = C''\tau \end{array} \qquad \textit{[general resolution]}$$

An analogous lifting lemma holds for factorization.

## Saturation of Sets of General Clauses

**Corollary 3.27** *Let $N$ be a set of general clauses saturated under Res, i. e., $Res(N) \subseteq N$. Then also $G_\Sigma(N)$ is saturated, that is,*

$$Res(G_\Sigma(N)) \subseteq G_\Sigma(N).$$

**Proof.** W.l.o.g. we may assume that clauses in $N$ are pairwise variable-disjoint. (Otherwise make them disjoint, and this renaming process changes neither $Res(N)$ nor $G_\Sigma(N)$.)

Let $C' \in Res(G_\Sigma(N))$, meaning (i) there exist resolvable ground instances $D\sigma$ and $C\rho$ of $N$ with resolvent $C'$, or else (ii) $C'$ is a factor of a ground instance $C\sigma$ of $C$.

Case (i): By the Lifting Lemma, $D$ and $C$ are resolvable with a resolvent $C''$ with $C''\tau = C'$, for a suitable substitution $\tau$. As $C'' \in N$ by assumption, we obtain that $C' \in G_\Sigma(N)$.

Case (ii): Similar. □

### Herbrand's Theorem

**Lemma 3.28** *Let $N$ be a set of $\Sigma$-clauses, let $\mathcal{A}$ be an interpretation. Then $\mathcal{A} \models N$ implies $\mathcal{A} \models G_\Sigma(N)$.*

**Lemma 3.29** *Let $N$ be a set of $\Sigma$-clauses, let $\mathcal{A}$ be a* Herbrand *interpretation. Then $\mathcal{A} \models G_\Sigma(N)$ implies $\mathcal{A} \models N$.*

**Theorem 3.30 (Herbrand)** *A set $N$ of $\Sigma$-clauses is satisfiable if and only if it has a Herbrand model over $\Sigma$.*

**Proof.** The "$\Leftarrow$" part is trivial. For the "$\Rightarrow$" part let $N \not\models \bot$.

$$
\begin{aligned}
N \not\models \bot &\Rightarrow \bot \notin Res^*(N) \qquad \text{(resolution is sound)}\\
&\Rightarrow \bot \notin G_\Sigma(Res^*(N))\\
&\Rightarrow G_\Sigma(Res^*(N))_{\mathcal{I}} \models G_\Sigma(Res^*(N)) \qquad \text{(Thm. 3.17; Cor. 3.27)}\\
&\Rightarrow G_\Sigma(Res^*(N))_{\mathcal{I}} \models Res^*(N) \qquad \text{(Lemma 3.29)}\\
&\Rightarrow G_\Sigma(Res^*(N))_{\mathcal{I}} \models N \qquad (N \subseteq Res^*(N)) \quad \Box
\end{aligned}
$$

### The Theorem of Löwenheim-Skolem

**Theorem 3.31 (Löwenheim–Skolem)** *Let $\Sigma$ be a countable signature and let $S$ be a set of closed $\Sigma$-formulas. Then $S$ is satisfiable iff $S$ has a model over a countable universe.*

**Proof.** If both $X$ and $\Sigma$ are countable, then $S$ can be at most countably infinite. Now generate, maintaining satisfiability, a set $N$ of clauses from $S$. This extends $\Sigma$ by at most countably many new Skolem functions to $\Sigma'$. As $\Sigma'$ is countable, so is $T_{\Sigma'}$, the universe of Herbrand-interpretations over $\Sigma'$. Now apply Theorem 3.30. □

### Refutational Completeness of General Resolution

**Theorem 3.32** *Let $N$ be a set of general clauses where $Res(N) \subseteq N$. Then*

$$N \models \bot \Leftrightarrow \bot \in N.$$

**Proof.** Let $Res(N) \subseteq N$. By Corollary 3.27: $Res(G_\Sigma(N)) \subseteq G_\Sigma(N)$

$$\begin{aligned} N \models \bot &\Leftrightarrow G_\Sigma(N) \models \bot && \text{(Lemma 3.28/3.29; Theorem 3.30)} \\ &\Leftrightarrow \bot \in G_\Sigma(N) && \text{(propositional resolution sound and complete)} \\ &\Leftrightarrow \bot \in N \quad \Box \end{aligned}$$

### Compactness of Predicate Logic

**Theorem 3.33 (Compactness Theorem for First-Order Logic)** *Let $S$ be a set of first-order formulas. $S$ is unsatisfiable iff some finite subset $S' \subseteq S$ is unsatisfiable.*

**Proof.** The "$\Leftarrow$" part is trivial. For the "$\Rightarrow$" part let $S$ be unsatisfiable and let $N$ be the set of clauses obtained by Skolemization and CNF transformation of the formulas in $S$. Clearly $Res^*(N)$ is unsatisfiable. By Theorem 3.32, $\bot \in Res^*(N)$, and therefore $\bot \in Res^n(N)$ for some $n \in \mathbb{N}$. Consequently, $\bot$ has a finite resolution proof $B$ of depth $\leq n$. Choose $S'$ as the subset of formulas in $S$ such that the corresponding clauses contain the assumptions (leaves) of $B$. $\Box$

## 3.11 First-Order Superposition with Selection

Motivation: Search space for *Res* *very* large.

Ideas for improvement:

1. In the completeness proof (Model Existence Theorem 2.13) one only needs to resolve and factor maximal atoms
   $\Rightarrow$ if the calculus is restricted to inferences involving maximal atoms, the proof remains correct
   $\Rightarrow$ *ordering restrictions*
2. In the proof, it does not really matter with which negative literal an inference is performed
   $\Rightarrow$ choose a negative literal don't-care-nondeterministically
   $\Rightarrow$ *selection*

### Selection Functions

A *selection function* is a mapping

$$\text{sel} : C \quad \mapsto \quad \text{set of occurrences of } \textit{negative} \text{ literals in } C$$

Example of selection with selected literals indicated as $\boxed{X}$:

$$\boxed{\neg A} \vee \neg A \vee B$$

$$\boxed{\neg B_0} \vee \boxed{\neg B_1} \vee A$$

Intuition:

- If a clause has at least one selected literal, compute only inferences that involve a selected literal.
- If a clause has no selected literals, compute only inferences that involve a maximal literal.

### Orderings for Terms, Atoms, Clauses

For first-order logic an ordering on the signature symbols is not sufficient to compare atoms, e.g., how to compare $P(a)$ and $P(b)$?

We propose the Knuth-Bendix Ordering for terms, atoms (with variables) which is then lifted as in the propositional case to literals and clauses.

## The Knuth-Bendix Ordering (Simple)

Let $\Sigma = (\Omega, \Pi)$ be a finite signature, let $\succ$ be a total ordering *("precedence")* on $\Omega \cup \Pi$, let $w : \Omega \cup \Pi \cup X \to \mathbb{R}^+$ be a *weight function*, satisfying $w(x) = w_0 \in \mathbb{R}^+$ for all variables $x \in X$ and $w(c) \geq w_0$ for all constants $c \in \Omega$.

The weight function $w$ can be extended to terms (atoms) as follows:

$$w(f(t_1, \dots, t_n)) = w(f) + \sum_{1 \leq i \leq n} w(t_i)$$

$$w(P(t_1, \dots, t_n)) = w(P) + \sum_{1 \leq i \leq n} w(t_i)$$

The *Knuth-Bendix ordering* $\succ_{\text{kbo}}$ on $\mathrm{T}_\Sigma(X)$ (atoms) induced by $\succ$ and $w$ is defined by: $s \succ_{\text{kbo}} t$ iff

(1) $\#(x, s) \geq \#(x, t)$ for all variables $x$ and $w(s) > w(t)$, or

(2) $\#(x, s) \geq \#(x, t)$ for all variables $x$, $w(s) = w(t)$, and

  (a) $s = f(s_1, \dots, s_m)$, $t = g(t_1, \dots, t_n)$, and $f \succ g$, or

  (b) $s = f(s_1, \dots, s_m)$, $t = f(t_1, \dots, t_m)$, and $(s_1, \dots, s_m)\ (\succ_{\text{kbo}})_{\text{lex}}\ (t_1, \dots, t_m)$.

where $\#(s, t) = |\{p \mid t|_p = s\}|$.

**Proposition 3.34** *The Knuth-Bendix ordering $\succ_{\text{kbo}}$ is*

*(1) a strict partial well-founded ordering on terms (atoms).*

*(2) stable under substitution: if $s \succ_{\text{kbo}} t$ then $s\sigma \succ_{\text{kbo}} t\sigma$ for any $\sigma$.*

*(3) total on ground terms (ground atoms).*

## Superposition Calculus $Sup_{\text{sel}}^{\succ}$

The resolution calculus $Sup_{\text{sel}}^{\succ}$ is parameterized by

- a selection function sel
- and a total and well-founded atom ordering $\succ$.

In the completeness proof, we talk about (strictly) maximal literals of *ground* clauses.

In the non-ground calculus, we have to consider those literals that correspond to (strictly) maximal literals of ground instances:

A literal $L$ is called *[strictly] maximal* in a clause $C$ if and only if there exists a ground substitution $\sigma$ such that $L\sigma$ is [strictly] maximal in $C\sigma$ (i.e., if for no other $L'$ in $C$: $L\sigma \prec L'\sigma$ [$L\sigma \preceq L'\sigma$]).

$$\frac{D \vee B \qquad C \vee \neg A}{(D \vee C)\sigma} \qquad [\textit{Superposition Left with Selection}]$$

if the following conditions are satisfied:

(i) $\sigma = \mathrm{mgu}(A, B)$;

(ii) $B\sigma$ strictly maximal in $D\sigma \vee B\sigma$;

(iii) nothing is selected in $D \vee B$ by sel;

(iv) either $\neg A$ is selected, or else nothing is selected in $C \vee \neg A$ and $\neg A\sigma$ is maximal in $C\sigma \vee \neg A\sigma$.

$$\frac{C \vee A \vee B}{(C \vee A)\sigma} \qquad [\textit{Factoring}]$$

if the following conditions are satisfied:

(i) $\sigma = \mathrm{mgu}(A, B)$;

(ii) $A\sigma$ is maximal in $C\sigma \vee A\sigma \vee B\sigma$;

(iii) nothing is selected in $C \vee A \vee B$ by sel.

### Special Case: Propositional Logic

For ground clauses the superposition inference rule simplifies to

$$\frac{D \vee P \qquad C \vee \neg P}{D \vee C}$$

if the following conditions are satisfied:

(i) $P \succ D$;

(ii) nothing is selected in $D \vee P$ by sel;

(iii) $\neg P$ is selected in $C \vee \neg P$, or else nothing is selected in $C \vee \neg P$ and $\neg P \succeq \max(C)$.

Note: For positive literals, $P \succ D$ is the same as $P \succ \max(D)$.

Analogously, the factoring rule simplifies to

$$\frac{C \vee P \vee P}{C \vee P}$$

if the following conditions are satisfied:

(i) $P$ is the largest literal in $C \vee P \vee P$;

(ii) nothing is selected in $C \vee P \vee P$ by sel.

### Search Spaces Become Smaller

| | | |
|---|---|---|
| 1 | $P \vee Q$ | |
| 2 | $P \vee \boxed{\neg Q}$ | |
| 3 | $\neg P \vee Q$ | |
| 4 | $\neg P \vee \boxed{\neg Q}$ | |
| 5 | $Q \vee Q$ | Res 1, 3 |
| 6 | $Q$ | Fact 5 |
| 7 | $\neg P$ | Res 6, 4 |
| 8 | $P$ | Res 6, 2 |
| 9 | $\bot$ | Res 8, 7 |

we assume $P \succ Q$ and sel as indicated by $\boxed{X}$. The maximal literal in a clause is depicted in red.

With this ordering and selection function the refutation proceeds strictly deterministically in this example. Generally, proof search will still be non-deterministic but the search space will be much smaller than with unrestricted resolution.

### Avoiding Rotation Redundancy

From

$$\dfrac{\dfrac{C_1 \vee P \quad C_2 \vee \neg P \vee Q}{C_1 \vee C_2 \vee Q} \qquad C_3 \vee \neg Q}{C_1 \vee C_2 \vee C_3}$$

we can obtain by *rotation*

$$\dfrac{C_1 \vee P \qquad \dfrac{C_2 \vee \neg P \vee Q \quad C_3 \vee \neg Q}{C_2 \vee \neg P \vee C_3}}{C_1 \vee C_2 \vee C_3}$$

another proof of the same clause. In large proofs many rotations are possible. However, if $P \succ Q$, then the second proof does not fulfill the orderings restrictions.

*Conclusion:* In the presence of orderings restrictions (however one chooses $\succ$) no rotations are possible. In other words, orderings identify exactly one representant in any class of rotation-equivalent proofs.

### Lifting Lemma for $Sup^{\succ}_{\text{sel}}$

**Lemma 3.35** *Let $D$ and $C$ be variable-disjoint clauses. If*

$$\begin{array}{cc} D & C \\ \downarrow \sigma & \downarrow \rho \\ \dfrac{D\sigma \qquad C\rho}{C'} & \end{array} \qquad \text{[propositional inference in } Sup^{\succ}_{\text{sel}}\text{]}$$

*and if* $\text{sel}(D\sigma) \simeq \text{sel}(D)$, $\text{sel}(C\rho) \simeq \text{sel}(C)$ *(that is, "corresponding" literals are selected), then there exists a substitution $\tau$ such that*

$$\begin{array}{c} \dfrac{D \qquad C}{C''} \\ \downarrow \tau \\ C' = C''\tau \end{array} \qquad \text{[inference in } Sup^{\succ}_{\text{sel}}\text{]}$$

An analogous lifting lemma holds for factorization.

### Saturation of General Clause Sets

**Corollary 3.36** *Let $N$ be a set of general clauses saturated under $Sup^{\succ}_{\text{sel}}$, i. e., $Sup^{\succ}_{\text{sel}}(N) \subseteq N$. Then there exists a selection function* $\text{sel}'$ *such that* $\text{sel}|_N = \text{sel}'|_N$ *and $G_\Sigma(N)$ is also saturated, i. e.,*

$$Sup^{\succ}_{\text{sel}'}(G_\Sigma(N)) \subseteq G_\Sigma(N).$$

**Proof.** We first define the selection function $\text{sel}'$ such that $\text{sel}'(C) = \text{sel}(C)$ for all clauses $C \in G_\Sigma(N) \cap N$. For $C \in G_\Sigma(N) \setminus N$ we choose a fixed but arbitrary clause $D \in N$ with $C \in G_\Sigma(D)$ and define $\text{sel}'(C)$ to be those occurrences of literals that are ground instances of the occurrences selected by sel in $D$. Then proceed as in the proof of Cor. 3.27 using the above lifting lemma. □

### Soundness and Refutational Completeness

**Theorem 3.37** *Let $\succ$ be an atom ordering and* sel *a selection function such that $Sup^{\succ}_{\text{sel}}(N) \subseteq N$. Then*

$$N \models \bot \Leftrightarrow \bot \in N$$

**Proof.** The "$\Leftarrow$" part is trivial. For the "$\Rightarrow$" part consider the propositional level: Construct a candidate interpretation $N_{\mathcal{I}}$ as for superposition without selection, except that clauses $C$ in $N$ that have selected literals are not productive, even when they are false in $N_C$ and when their maximal atom occurs only once and positively. The result then follows by Corollary 3.36. □

### Craig-Interpolation

A theoretical application of superposition is Craig-Interpolation:

**Theorem 3.38 (Craig 1957)** *Let $\phi$ and $\psi$ be two propositional formulas such that $\phi \models \psi$. Then there exists a formula $\chi$ (called the* interpolant *for $\phi \models \psi$), such that $\chi$ contains only prop. variables occurring both in $\phi$ and in $\psi$, and such that $\phi \models \chi$ and $\chi \models \psi$.*

**Proof.** Translate $\phi$ and $\neg\psi$ into CNF. let $N$ and $M$, resp., denote the resulting clause set. Choose an atom ordering $\succ$ for which the prop. variables that occur in $\phi$ but not in $\psi$ are maximal. Saturate $N$ into $N^*$ w. r. t. $Sup_{\text{sel}}^{\succ}$ with an empty selection function sel . Then saturate $N^* \cup M$ w. r. t. $Sup_{\text{sel}}^{\succ}$ to derive $\bot$. As $N^*$ is already saturated, due to the ordering restrictions only inferences need to be considered where premises, if they are from $N^*$, only contain symbols that also occur in $\psi$. The conjunction of these premises is an interpolant $\chi$. The theorem also holds for first-order formulas. For universal formulas the above proof can be easily extended. In the general case, a proof based on superposition technology is more complicated because of Skolemization. $\square$

### Redundancy

So far: local restrictions of the resolution inference rules using orderings and selection functions.

Is it also possible to delete clauses altogether? Under which circumstances are clauses unnecessary? (Conjecture: e. g., if they are tautologies or if they are subsumed by other clauses.)

Intuition: If a clause is guaranteed to be neither a minimal counterexample nor productive, then we do not need it.

### A Formal Notion of Redundancy

Let $N$ be a set of ground clauses and $C$ a ground clause (not necessarily in $N$). $C$ is called *redundant* w. r. t. $N$, if there exist $C_1, \ldots, C_n \in N$, $n \geq 0$, such that $C_i \prec C$ and $C_1, \ldots, C_n \models C$.

Redundancy for general clauses: $C$ is called *redundant* w. r. t. $N$, if all ground instances $C\sigma$ of $C$ are redundant w. r. t. $G_\Sigma(N)$.

Intuition: Redundant clauses are neither minimal counterexamples nor productive.

Note: The same ordering $\prec$ is used for ordering restrictions and for redundancy (and for the completeness proof).