2.10 Superposition Versus CDCL

We will establish a relationship between Superposition and CDCL operating on a clause set N:

Superposition: Is based on an ordering $\prec$. It computes a model assumption $N_{\mathcal{I}}$. Either $N_{\mathcal{I}}$ is a model, N contains the empty clause, or there is an inference on the minimal false clause with respect to $\prec$.

CDCL: Is based on a variable selection heuristic. It computes a model assumption via decision variables and propagation. Either this assumption is a model of N, N contains the empty clause, or there is a backjump clause that is learned.

Proposition 2.20 *Let $(L_1 + L_2 + \ldots + L_k; N)$ be a CDCL with eager propagation state. Some of the L_i may be decision literals and the corresponding propositional variables are $P_1, \ldots, P_k$. Furthermore, let us assume that $L_1 + \ldots + L_{k-1}$ is a partial valuation that does not falsify any clause in N whereas $L_1 + L_2 + \ldots + L_k$ falsifies some clause $C \vee \overline{L_k} \in N$. Then*

(a) L_k is a propagated literal.

(b) The resolvent between $C \vee \overline{L_k}$ and the clause propagating L_k is a superposition inference and the conclusion is not redundant with respect to the ordering $P_1 \prec P_2 \ldots \prec P_k$.

Proof. (a) The clause $C \vee \overline{L_k}$ propagates $\overline{L_k}$ with respect to $L_1 + \ldots + L_{k-1}$, so with eager propagation, the literal L_k cannot be decision literal but was propagated by a clause $C' \vee L_k \in N$.

(b) Both C and C' only contain literals with variables from $P_1, \ldots, P_{k-1}$. Since we assume duplicate literals to be removed and tautologies to be deleted, the literal $\overline{L_k}$ is strictly maximal in $C \vee \overline{L_k}$ and L_k is strictly maximal in $C' \vee L_k$. So resolving on L_k is a superposition inference with respect to the variable ordering $P_1 \prec P_2 \ldots \prec P_k$. Now assume $C \vee C'$ is redundant, i.e., there are clauses $D_1, \ldots, D_n$ from N with $D_i \prec C \vee C'$ and $D_1, \ldots, D_n \models C \vee C'$. Since $C \vee C'$ is false in $L_1 + \ldots + L_{k-1}$ there is at least one D_i that is also false in $L_1 + \ldots + L_{k-1}$. A contradiction against the assumption that $L_1 + \ldots + L_{k-1}$ does not falsify any clause in N. □

Proposition 2.21 *The 1UIP backjump clause is not redundant.*

Proof. By Proposition 2.20 a one resolution step 1UIP backjump clause has this property. The argument in the proof of Proposition 2.20 can be repeated until we reach the first decision literal L_m by resolving away $L_k, L_{k-1}, \ldots, L_{m+1}$. □

Proposition 2.22 *Let $(L_1 + L_2 + \ldots + L_k; N)$ be a CDCL with eager propagation state. We assume that all decision literals among the L_i are negative and let the corresponding propositional variables be $P_1, \ldots, P_k$. Furthermore, let us assume that $L_1 + \ldots + L_k$ is a partial valuation that does not falsify any clause in N. Then $N_{\mathcal{I}}^{\prec P_{k+1}} = \{P_1, \ldots, P_k\} \cap \{L_1, \ldots, L_k\}$ with ordering $P_1 \prec P_2 \ldots \prec P_{k+1}$.*

Proof. We assume that there is a variable $P_{k+1} \in \Sigma$ for otherwise it can be added. By induction on k. For the base case $k = 1$ we distinguish two cases. If L_1 is propagated then there is a clause $L_1 \in N$. In case L_1 is positive then it is also productive and $L_1 \in N_{\mathcal{I}}^{\prec P_2}$. If it is negative then there cannot be a clause $P_1 \in N$, so $P_1 \notin N_{\mathcal{I}}^{\prec P_2}$.

For the induction step assume $N_{\mathcal{I}}^{\prec P_k} = \{P_1, \ldots, P_{k-1}\} \cap \{L_1, \ldots, L_{k-1}\}$. If L_k is propagated and positive, then there is a clause $C \vee L_k$ where all atoms in C are from $\{P_1, \ldots, P_{k-1}\}$ and hence L_k is strictly maximal in $C \vee L_k$, the clause C is false in $N_{\mathcal{I}}^{\prec P_k}$ and therefore L_k is produced, proving $N_{\mathcal{I}}^{\prec P_{k+1}} = \{P_1, \ldots, P_k\} \cap \{L_1, \ldots, L_k\}$.

If L_k is propagated and negative, then there cannot be a clause $C \vee P_k \in N^{\prec P_{k+1}}$ with C false in $N_{\mathcal{I}}^{\prec P_k}$, because for otherwise $L_1 + \ldots + L_k$ falsifies a clause in N. So there is no clause in N producing P_k and hence $N_{\mathcal{I}}^{\prec P_{k+1}} = \{P_1, \ldots, P_k\} \cap \{L_1, \ldots, L_k\}$.

If L_k is a decision literal and therefore negative, there cannot be a clause $C \vee P_k \in N^{\prec P_{k+1}}$ with C false in $N_{\mathcal{I}}^{\prec P_k}$, because we assume eager propagation and so again $N_{\mathcal{I}}^{\prec P_{k+1}} = \{P_1, \ldots, P_k\} \cap \{L_1, \ldots, L_k\}$. □

3 First-Order Logic

First-order logic

- formalizes fundamental mathematical concepts
- is expressive (Turing-complete)
- is not too expressive (e. g. not axiomatizable: natural numbers, uncountable sets)
- has a rich structure of decidable fragments
- has a rich model and proof theory

First-order logic is also called (first-order) *predicate logic*.