If a unifier of E is more general than any other unifier of E, then we speak of a *most general unifier* of E, denoted by $\mathrm{mgu}(E)$.

Proposition 3.20

(i) $\leq$ *is a quasi-ordering on substitutions, and* $\circ$ *is associative.*

(ii) If $\sigma \leq \tau$ *and* $\tau \leq \sigma$ *(we write* $\sigma \sim \tau$ *in this case), then* $x\sigma$ *and* $x\tau$ *are equal up to (bijective) variable renaming, for any* x *in* X*.*

A substitution σ is called *idempotent*, if $\sigma \circ \sigma = \sigma$.

Proposition 3.21 *σ is idempotent iff $dom(\sigma) \cap codom(\sigma) = \emptyset$.*

Rule-Based Naive Standard Unification

$$
\begin{array}{rcl}
t \doteq t, E & \Rightarrow_{SU} & E \\
f(s_1, \ldots, s_n) \doteq f(t_1, \ldots, t_n), E & \Rightarrow_{SU} & s_1 \doteq t_1, \ldots, s_n \doteq t_n, E \\
f(\ldots) \doteq g(\ldots), E & \Rightarrow_{SU} & \bot \\
x \doteq t, E & \Rightarrow_{SU} & x \doteq t, E\{x \mapsto t\} \\
 & & \text{if } x \in var(E), x \notin var(t) \\
x \doteq t, E & \Rightarrow_{SU} & \bot \\
 & & \text{if } x \neq t, x \in var(t) \\
t \doteq x, E & \Rightarrow_{SU} & x \doteq t, E \\
 & & \text{if } t \notin X
\end{array}
$$

SU: Main Properties

If $E = x_1 \doteq u_1, \ldots, x_k \doteq u_k$, with x_i pairwise distinct, $x_i \notin var(u_j)$, then E is called an (equational problem in) *solved form* representing the solution $\sigma_E = \{x_1 \mapsto u_1, \ldots, x_k \mapsto u_k\}$.

Proposition 3.22 *If E is a solved form then σ_E is an mgu of E.*

Theorem 3.23

1. *If $E \Rightarrow_{SU} E'$ then σ is a unifier of E iff σ is a unifier of E'*
2. *If $E \Rightarrow^*_{SU} \bot$ then E is not unifiable.*
3. *If $E \Rightarrow^*_{SU} E'$ with E' in solved form, then $\sigma_{E'}$ is an mgu of E.*

Proof. (1) We have to show this for each of the rules. Let's treat the case for the 4th rule here. Suppose σ is a unifier of $x \doteq t$, that is, $x\sigma = t\sigma$. Thus, $\sigma \circ \{x \mapsto t\} = \sigma[x \mapsto t\sigma] = \sigma[x \mapsto x\sigma] = \sigma$. Therefore, for any equation $u \doteq v$ in E: $u\sigma = v\sigma$, iff $u\{x \mapsto t\}\sigma = v\{x \mapsto t\}\sigma$. (2) and (3) follow by induction from (1) using Proposition 3.22. □

Main Unification Theorem

Theorem 3.24 *E is unifiable if and only if there is a most general unifier σ of E, such that σ is idempotent and $dom(\sigma) \cup codom(\sigma) \subseteq var(E)$.*

Proof.

- $\Rightarrow_{SU}$ is Noetherian. A suitable lexicographic ordering on the multisets E (with $\bot$ minimal) shows this. Compare in this order:
    1. the number of defined variables (d.h. variables x in equations $x \doteq t$ with $x \notin var(t)$), which also occur outside their definition elsewhere in E;
    2. the multiset ordering induced by (i) the size (number of symbols) in an equation; (ii) if sizes are equal consider $x \doteq t$ smaller than $t \doteq x$, if $t \notin X$.
- A system E that is irreducible w. r. t. $\Rightarrow_{SU}$ is either $\bot$ or a solved form.
- Therefore, reducing any E by SU will end (no matter what reduction strategy we apply) in an irreducible E' having the same unifiers as E, and we can read off the mgu (or non-unifiability) of E from E' (Theorem 3.23, Proposition 3.22).
- σ is idempotent because of the substitution in rule 4. $dom(\sigma) \cup codom(\sigma) \subseteq var(E)$, as no new variables are generated.

□

Rule-Based Polynomial Unification

Problem: using $\Rightarrow_{SU}$, an *exponential growth* of terms is possible.

The following unification algorithm avoids this problem, at least if the final solved form is represented as a DAG.

$$
\begin{array}{rcl}
t \doteq t, E & \Rightarrow_{PU} & E \\
f(s_1, \ldots, s_n) \doteq f(t_1, \ldots, t_n), E & \Rightarrow_{PU} & s_1 \doteq t_1, \ldots, s_n \doteq t_n, E \\
f(\ldots) \doteq g(\ldots), E & \Rightarrow_{PU} & \bot \\
x \doteq y, E & \Rightarrow_{PU} & x \doteq y, E\{x \mapsto y\} \\
 & & \text{if } x \in var(E), x \neq y \\
x_1 \doteq t_1, \ldots, x_n \doteq t_n, E & \Rightarrow_{PU} & \bot \\
 & & \text{if there are positions } p_i \text{ with} \\
 & & t_i/p_i = x_{i+1}, t_n/p_n = x_1 \\
 & & \text{and some } p_i \neq \epsilon \\
x \doteq t, E & \Rightarrow_{PU} & \bot \\
 & & \text{if } x \neq t, x \in var(t) \\
t \doteq x, E & \Rightarrow_{PU} & x \doteq t, E \\
 & & \text{if } t \notin X \\
x \doteq t, x \doteq s, E & \Rightarrow_{PU} & x \doteq t, t \doteq s, E \\
 & & \text{if } t, s \notin X \text{ and } |t| \leq |s|
\end{array}
$$

Properties of PU

Theorem 3.25

1. *If* $E \Rightarrow_{PU} E'$ *then* σ *is a unifier of* E *iff* σ *is a unifier of* E'
2. *If* $E \Rightarrow^{*}_{PU} \bot$ *then* E *is not unifiable.*
3. *If* $E \Rightarrow^{*}_{PU} E'$ *with* E' *in solved form, then* $\sigma_{E'}$ *is an mgu of* E.

Note: The solved form of $\Rightarrow_{PU}$ is different form the solved form obtained from $\Rightarrow_{SU}$. In order to obtain the unifier $\sigma_{E'}$, we have to sort the list of equality problems $x_i \doteq t_i$ in such a way that x_i does not occur in t_j for $j < i$, and then we have to compose the substitutions $\{x_1 \mapsto t_1\} \circ \cdots \circ \{x_k \mapsto t_k\}$.

Lifting Lemma

Lemma 3.26 *Let* C *and* D *be variable-disjoint clauses. If*

$$\begin{array}{ccc} D & & C \\ \downarrow \sigma & & \downarrow \rho \\ \dfrac{D\sigma \qquad}{} & \hspace{-2em} & \dfrac{\qquad C\rho}{} \end{array} \qquad \frac{D\sigma \qquad C\rho}{C'} \quad \text{[propositional resolution]}$$

then there exists a substitution τ *such that*

$$\frac{D \qquad C}{C''} \quad \text{[general resolution]}$$
$$\downarrow \tau$$
$$C' = C''\tau$$

An analogous lifting lemma holds for factorization.

Saturation of Sets of General Clauses

Corollary 3.27 *Let* N *be a set of general clauses saturated under* Res, *i. e.,* $Res(N) \subseteq N$. *Then also* $G_\Sigma(N)$ *is saturated, that is,*

$$Res(G_\Sigma(N)) \subseteq G_\Sigma(N).$$