2.2 Semantics

In *classical logic* (dating back to Aristoteles) there are “only” two truth values “true” and “false” which we shall denote, respectively, by 1 and 0.

There are *multi-valued logics* having more than two truth values.

Valuations

A propositional variable has no intrinsic meaning. The meaning of a propositional variable has to be defined by a valuation.

A Σ-*valuation* is a map

$$\mathcal{A} : \Sigma \to \{0, 1\}.$$

where $\{0, 1\}$ is the set of *truth values*.

Truth Value of a Formula in $\mathcal{A}$

Given a Σ-valuation $\mathcal{A}$, the function can be extened to $\mathcal{A} : \mathrm{PROP}(\Sigma) \to \{0, 1\}$ by:

$$\begin{aligned}
\mathcal{A}(\bot) &= 0 \\
\mathcal{A}(\top) &= 1 \\
\mathcal{A}(\neg\phi) &= 1 - \mathcal{A}(\phi) \\
\mathcal{A}(\phi \land \psi) &= \min(\{\mathcal{A}(\phi), \mathcal{A}(\psi)\}) \\
\mathcal{A}(\phi \lor \psi) &= \max(\{\mathcal{A}(\phi), \mathcal{A}(\psi)\}) \\
\mathcal{A}(\phi \to \psi) &= \max(\{(1 - \mathcal{A}(\phi)), \mathcal{A}(\psi)\}) \\
\mathcal{A}(\phi \leftrightarrow \psi) &= \text{if } \mathcal{A}(\phi) = \mathcal{A}(\psi) \text{ then } 1 \text{ else } 0
\end{aligned}$$

2.3 Models, Validity, and Satisfiability

ϕ is *valid* in $\mathcal{A}$ ($\mathcal{A}$ is a *model* of ϕ; ϕ *holds* under $\mathcal{A}$):

$$\mathcal{A} \models \phi \;:\Leftrightarrow\; \mathcal{A}(\phi) = 1$$

ϕ is *valid* (or is a *tautology*):

$$\models \phi \;:\Leftrightarrow\; \mathcal{A} \models \phi \text{ for all } \Sigma\text{-valuations } \mathcal{A}$$

ϕ is called *satisfiable* if there exists an $\mathcal{A}$ such that $\mathcal{A} \models \phi$. Otherwise ϕ is called *unsatisfiable* (or *contradictory*).

Entailment and Equivalence

ϕ *entails* (*implies*) ψ (or ψ *is a consequence of* ϕ), written $\phi \models \psi$, if for all Σ-valuations $\mathcal{A}$ we have $\mathcal{A} \models \phi \ \Rightarrow \ \mathcal{A} \models \psi$.

ϕ and ψ are called *equivalent*, written $\phi \models\!\!\!\!\!\!\!\!\dashv\;\; \psi$, if for all Σ-valuations $\mathcal{A}$ we have $\mathcal{A} \models \phi \ \Leftrightarrow \ \mathcal{A} \models \psi$.

Proposition 2.3 *$\phi \models \psi$ if and only if $\models (\phi \rightarrow \psi)$.*

Proof. ($\Rightarrow$) Suppose that ϕ entails ψ. Let $\mathcal{A}$ be an arbitrary Σ-valuation. We have to show that $\mathcal{A} \models \phi \rightarrow \psi$. If $\mathcal{A}(\phi) = 1$, then $\mathcal{A}(\psi) = 1$ (since $\phi \models \psi$), and hence $\mathcal{A}(\phi \rightarrow \psi) = 1$. Otherwise if $\mathcal{A}(\phi) = 0$, then $\mathcal{A}(\phi \rightarrow \psi) = \max(\{1, \mathcal{A}(\psi)\}) = 1$ independently of $\mathcal{A}(\psi)$. In both cases, $\mathcal{A} \models \phi \rightarrow \psi$.

($\Leftarrow$) Suppose that ϕ does not entail ψ. Then there exists a Σ-valuation $\mathcal{A}$ such that $\mathcal{A} \models \phi$, but not $\mathcal{A} \models \psi$. Consequently, $\mathcal{A}(\phi \rightarrow \psi) = \max(\{(1 - \mathcal{A}(\phi)), \mathcal{A}(\psi)\}) = \max(\{0, 0\}) = 0$, so $(\phi \rightarrow \psi)$ does not hold in $\mathcal{A}$. □

Proposition 2.4 *$\phi \models\!\!\!\!\!\!\!\!\dashv\;\; \psi$ if and only if $\models (\phi \leftrightarrow \psi)$.*

Proof. Analogously to Prop. 2.3. □

Entailment is extended to sets of formulas N in the "natural way":

$N \models \phi$ if for all Σ-valuations $\mathcal{A}$:
if $\mathcal{A} \models \psi$ for all $\psi \in N$, then $\mathcal{A} \models \phi$.

Note: formulas are always finite objects; but sets of formulas may be infinite. Therefore, it is in general not possible to replace a set of formulas by the conjunction of its elements.

Validity vs. Unsatisfiability

Validity and unsatisfiability are just two sides of the same medal as explained by the following proposition.

Proposition 2.5 *ϕ is valid if and only if $\neg\phi$ is unsatisfiable.*

Proof. ($\Rightarrow$) If ϕ is valid, then $\mathcal{A}(\phi) = 1$ for every valuation $\mathcal{A}$. Hence $\mathcal{A}(\neg\phi) = (1 - \mathcal{A}(\phi)) = 0$ for every valuation $\mathcal{A}$, so $\neg\phi$ is unsatisfiable.

($\Leftarrow$) Analogously. □

Hence in order to design a theorem prover (validity checker) it is sufficient to design a checker for unsatisfiability.

In a similar way, entailment $N \models \phi$ can be reduced to unsatisfiability:

Proposition 2.6 *$N \models \phi$ if and only if $N \cup \{\neg\phi\}$ is unsatisfiable.*

Checking Unsatisfiability

Every formula ϕ contains only finitely many propositional variables. Obviously, $\mathcal{A}(\phi)$ depends only on the values of those finitely many variables in ϕ under $\mathcal{A}$.

If ϕ contains n distinct propositional variables, then it is sufficient to check 2^n valuations to see whether ϕ is satisfiable or not.
$\Rightarrow$ truth table.

So the satisfiability problem is clearly deciadable (but, by Cook's Theorem, NP-complete).

Nevertheless, in practice, there are (much) better methods than truth tables to check the satisfiability of a formula. (later more)

Truth Table

Let ϕ be a propositional formula over variables $P_1, \ldots, P_n$ and $k = |\operatorname{pos}(\phi)|$. Then a *complete truth table* for ϕ is a table with $n + k$ columns and $2^n + 1$ rows of the form

P_1	$\ldots$	P_n	$\phi\|_{p_1}$	$\ldots$	$\phi\|_{p_k}$
0	$\ldots$	0	$\mathcal{A}_1(\phi\|_{p_1})$	$\ldots$	$\mathcal{A}_1(\phi\|_{p_k})$
			$\vdots$		
1	$\ldots$	1	$\mathcal{A}_{2^n}(\phi\|_{p_1})$	$\ldots$	$\mathcal{A}_{2^n}(\phi\|_{p_k})$

such that the $\mathcal{A}_i$ are exactly the 2^n different valuations for $P_1, \ldots, P_n$ and either $p_i \parallel p_{i+j}$ or $p_i \geq p_{i+j}$, in particular $p_k = \epsilon$ and $\phi|_{p_k} = \phi$ for all $i, j \geq 0$, $i + j \leq k$.

Truth tables can be used to check validity, satisfiablity or unsatisfiability of a formula in a systematic way.

They have the nice property that if the rows are filled from left to right, then in order to compute $\mathcal{A}_i(\phi|_{p_j})$ the values for $\mathcal{A}_i$ of $\phi|_{p_jh}$ are already computed, $h \in \{1, 2\}$.