6.3 Reduction Pairs and Argument Filterings

Goal: Show the non-existence of K-minimal infinite rewrite sequences

$$t_1 \rightarrow_R^* u_1 \rightarrow_K t_2 \rightarrow_R^* u_2 \rightarrow_K \dots$$

using well-founded orderings.

We observe that the requirements for the orderings used here are less restrictive than for reduction orderings:

K-rules are only used at the top, so we need stability under substitutions, but compatibility with contexts is unnecessary.

While $\rightarrow_K$-steps should be decreasing, for $\rightarrow_R$-steps it would be sufficient to show that they are not increasing.

This motivates the following definitions:

Rewrite quasi-ordering $\succsim$:

reflexive and transitive binary relation, stable under substitutions, compatible with contexts.

Reduction pair $(\succsim, \succ)$:

$\succsim$ is a rewrite quasi-ordering.

$\succ$ is a well-founded ordering that is stable under substitutions.

$\succsim$ and $\succ$ are compatible: $\succsim \circ \succ \subseteq \succ$ or $\succ \circ \succsim \subseteq \succ$.

(In practice, $\succ$ is almost always the strict part of the quasi-ordering $\succsim$.)

Clearly, for any reduction ordering $\succ$, $(\succeq, \succ)$ is a reduction pair. More general reduction pairs can be obtained using argument filterings:

Argument filtering π:

$\pi : \Omega \cup \Omega^\sharp \rightarrow \mathbb{N} \cup \text{list of } \mathbb{N}$

$$\pi(f) = \begin{cases} i \in \{1, \dots, \mathit{arity}(f)\}, \text{ or} \\ [i_1, \dots, i_k], \text{ where } 1 \le i_1 < \dots < i_k \le \mathit{arity}(f),\ 0 \le k \le \mathit{arity}(f) \end{cases}$$

Extension to terms:

$\pi(x) = x$

$\pi(f(t_1, \dots, t_n)) = \pi(t_i)$, if $\pi(f) = i$

$\pi(f(t_1, \dots, t_n)) = f'(\pi(t_{i_1}), \dots, \pi(t_{i_k}))$, if $\pi(f) = [i_1, \dots, i_k]$,
where f'/k is a new function symbol.

Let $\succ$ be a reduction ordering, let π be an argument filtering. Define $s \succ_\pi t$ iff $\pi(s) \succ \pi(t)$ and $s \succsim_\pi t$ iff $\pi(s) \succeq \pi(t)$.

Lemma 6.2 *$(\succsim_\pi, \succ_\pi)$ is a reduction pair.*

Proof. Follows from the following two properties:

$\pi(s\sigma) = \pi(s)\sigma_\pi$, where σ_π is the substitution that maps x to $\pi(\sigma(x))$.

$$\pi(s[u]_p) = \begin{cases} \pi(s), & \text{if } p \text{ does not correspond to any position in } \pi(s) \\ \pi(s)[\pi(u)]_q, & \text{if } p \text{ corresponds to } q \text{ in } \pi(s) \end{cases}$$ □

For interpretation-based orderings (such as polynomial orderings) the idea of "cutting out" certain subterms can be included directly in the definition of the ordering:

Reduction pairs by interpretation:

Let $\mathcal{A}$ be a Σ-algebra; let $\succ$ be a well-founded strict partial ordering on its universe.

Assume that all interpretations $f_\mathcal{A}$ of function symbols are *weakly monotone*, i. e., $a_i \succeq b_i$ implies $f(a_1, \dots, , a_n) \succeq f(b_1, \dots, b_n)$ for all $a_i, b_i \in U_\mathcal{A}$.

Define $s \succsim_\mathcal{A} t$ iff $\mathcal{A}(\beta)(s) \succeq \mathcal{A}(\beta)(t)$ for all assignments $\beta : X \to U_\mathcal{A}$; define $s \succ_\mathcal{A} t$ iff $\mathcal{A}(\beta)(s) \succ \mathcal{A}(\beta)(t)$ for all assignments $\beta : X \to U_\mathcal{A}$.

Then $(\succsim_\mathcal{A}, \succ_\mathcal{A})$ is a reduction pair.

For polynomial orderings, this definition permits interpretations of function symbols where some variable does not occur at all (e. g., $P_f(X, Y) = 2X + 1$ for a *binary* function symbol). It is no longer required that *every* variable must occur with some positive coefficient.

Theorem 6.3 (Arts and Giesl) *Let K be a cycle in the dependency graph of the TRS R. If there is a reduction pair $(\succsim, \succ)$ such that*

- *$l \succsim r$ for all $l \to r \in R$,*
- *$l \succsim r$ or $l \succ r$ for all $l \to r \in K$,*
- *$l \succ r$ for at least one $l \to r \in K$,*

then there is no K-minimal infinite sequence.

Proof. Assume that

$$t_1 \to_R^* u_1 \to_K t_2 \to_R^* u_2 \to_K \dots$$

is a K-minimal infinite rewrite sequence.

As $l \succsim r$ for all $l \to r \in R$, we obtain $t_i \succsim u_i$ by stability under substitutions, compatibility with contexts, reflexivity and transitivity.

As $l \succsim r$ or $l \succ r$ for all $l \to r \in K$, we obtain $u_i \ (\succsim \cup \succ) \ t_{i+1}$ by stability under substitutions.

So we get an infinite $(\succsim \cup \succ)$-sequence containing infinitely many $\succ$-steps (since every DP in K, in particular the one for which $l \succ r$ holds, is used infinitely often).

By compatibility of $\succsim$ and $\succ$, we can transform this into an infinite $\succ$-sequence, contradicting well-foundedness. $\square$

The idea can be extended to SCCs in the same way as for the subterm criterion:

Search for a reduction pair $(\succsim, \succ)$ such that $l \succsim r$ for all $l \to r \in R$ and $l \succsim r$ or $l \succ r$ for all DPs $l \to r$ in the SCC. Delete all DPs in the SCC for which $l \succ r$. Then re-compute SCCs for the remaining graph and re-start.

Example: Consider the following TRS R from [Arts and Giesl]:

$$
\begin{aligned}
&minus(x, 0) \to x && (1)\\
&minus(s(x), s(y)) \to minus(x, y) && (2)\\
&quot(0, s(y)) \to 0 && (3)\\
&quot(s(x), s(y)) \to s(quot(minus(x, y), s(y))) && (4)
\end{aligned}
$$

(R is not contained in any simplification ordering, since the left-hand side of rule (4) is embedded in the right-hand side after instantiating y by $s(x)$.)

R has three dependency pairs:

$$
\begin{aligned}
&minus^\sharp(s(x), s(y)) \to minus^\sharp(x, y) && (5)\\
&quot^\sharp(s(x), s(y)) \to quot^\sharp(minus(x, y), s(y)) && (6)\\
&quot^\sharp(s(x), s(y)) \to minus^\sharp(x, y) && (7)
\end{aligned}
$$

The dependency graph of R is

$$
\begin{tikzcd}
(5) \arrow[loop below] & (7) \arrow[l] & (6) \arrow[l] \arrow[loop below]
\end{tikzcd}
$$

There are exactly two SCCs (and also two cycles). The cycle at (5) can be handled using the subterm criterion with $\pi(minus^\sharp) = 1$. For the cycle at (6) we can use an argument filtering π that maps *minus* to 1 and leaves all other function symbols unchanged (that is, $\pi(g) = [1, \ldots, arity(g)]$ for every g different from *minus*.) After applying the argument filtering, we compare left and right-hand sides using an LPO with precedence $quot > s$ (the precedence of other symbols is irrelevant). We obtain $l \succ r$ for (6) and $l \succsim r$ for (1), (2), (3), (4), so the previous theorem can be applied.

The methods described so far are particular cases of *DP processors:*

A DP processor

$$\frac{(G, R)}{(G_1, R_1),\ \ \ldots,\ \ (G_n, R_n)}$$

takes a graph G and a TRS R as input and produces a set of pairs consisting of a graph and a TRS.

It is sound and complete if there are K-minimal infinite sequences for G and R if and only if there are K-minimal infinite sequences for at least one of the pairs (G_i, R_i).

Examples:

$$\frac{(G, R)}{(SCC_1, R),\ \ \ldots,\ \ (SCC_n, R)}$$

where $SCC_1, \ldots, SCC_n$ are the strongly connected components of G.

$$\frac{(G, R)}{(G \setminus N, R)}$$

if there is an SCC of G and a simple projection π such that $\pi(l) \trianglerighteq \pi(r)$ for all DPs $l \to r$ in the SCC, and N is the set of DPs of the SCC for which $\pi(l) \vartriangleright \pi(r)$.

(and analogously for reduction pairs)

The dependency method can also be used for proving termination of *innermost rewriting:* $s \xrightarrow{\text{i}}_R t$ if $s \to_R t$ at position p and no rule of R can be applied at a position strictly below p. (DP processors for innermost termination are more powerful than for ordinary termination, and for program analysis, innermost termination is usually sufficient.)