3.6 Getting Small Skolem Functions

A clause set that is better suited for automated theorem proving can be obtained using the following steps:

- produce a negation normal form (NNF)
- apply miniscoping
- rename all variables
- skolemize

Negation Normal Form (NNF)

Apply the rewrite relation $\Rightarrow_{NNF}$, where F is the overall formula:

$$\begin{array}{rcl}
G \leftrightarrow H & \Rightarrow_{NNF} & (G \rightarrow H) \wedge (H \rightarrow G) \\
& & \text{ if } F/p = G \leftrightarrow H \text{ has positive polarity in } F \\
G \leftrightarrow H & \Rightarrow_{NNF} & (G \wedge H) \vee (\neg H \wedge \neg G) \\
& & \text{ if } F/p = G \leftrightarrow H \text{ has negative polarity in } F \\
\neg Qx\, G & \Rightarrow_{NNF} & \overline{Q}x \neg G \\
\neg(G \vee H) & \Rightarrow_{NNF} & \neg G \wedge \neg H \\
\neg(G \wedge H) & \Rightarrow_{NNF} & \neg G \vee \neg H \\
G \rightarrow H & \Rightarrow_{NNF} & \neg G \vee H \\
\neg\neg G & \Rightarrow_{NNF} & G
\end{array}$$

Miniscoping

Apply the rewrite relation $\Rightarrow_{MS}$. For the below rules we assume that x occurs freely in G, H, but x does not occur freely in F:

$$\begin{array}{rcl}
Qx\,(G \wedge F) & \Rightarrow_{MS} & Qx\, G \wedge F \\
Qx\,(G \vee F) & \Rightarrow_{MS} & Qx\, G \vee F \\
\forall x\,(G \wedge H) & \Rightarrow_{MS} & \forall x\, G \wedge \forall x\, H \\
\exists x\,(G \vee H) & \Rightarrow_{MS} & \exists x\, G \vee \exists x\, H
\end{array}$$

Variable Renaming

Rename all variables in F such that there are no two different positions p, q with $F/p = Qx\, G$ and $F/q = Q'x\, H$.

Standard Skolemization

Let F be the overall formula, then apply the rewrite rule:

$$\begin{array}{lll} \exists x\, H & \Rightarrow_{SK} & H\{x \mapsto f(y_1, \ldots, y_n)\} \\ & & \text{if } F/p = \exists x\, H \text{ and } p \text{ has minimal length,} \\ & & \{y_1, \ldots, y_n\} \text{ are the free variables in } \exists x\, H, \\ & & f/n \text{ is a new function symbol} \end{array}$$

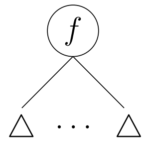
3.7 Herbrand Interpretations

From now on we shall consider FOL without equality. We assume that Ω contains at least one constant symbol.

A *Herbrand interpretation* (over Σ) is a Σ-algebra $\mathcal{A}$ such that

- $U_{\mathcal{A}} = \mathrm{T}_\Sigma$ (= the set of ground terms over Σ)
- $f_{\mathcal{A}} : (s_1, \ldots, s_n) \mapsto f(s_1, \ldots, s_n),\ f/n \in \Omega$

$$f_{\mathcal{A}}(\triangle, \ldots, \triangle) =$$



In other words, *values are fixed* to be ground terms and *functions are fixed* to be the *term constructors*. Only predicate symbols $P/m \in \Pi$ may be freely interpreted as relations $P_{\mathcal{A}} \subseteq \mathrm{T}_\Sigma^m$.

Proposition 3.12 *Every set of ground atoms I uniquely determines a Herbrand interpretation $\mathcal{A}$ via*

$$(s_1, \ldots, s_n) \in P_{\mathcal{A}} \quad :\Leftrightarrow \quad P(s_1, \ldots, s_n) \in I$$

Thus we shall identify Herbrand interpretations (over Σ) with sets of Σ-ground atoms.

Example: $\Sigma_{Pres} = (\{0/0, s/1, +/2\},\ \{</2, \leq/2\})$

$\mathbb{N}$ as Herbrand interpretation over Σ_{Pres}:

$$
\begin{aligned}
I = \{ \quad & 0 \leq 0,\ 0 \leq s(0),\ 0 \leq s(s(0)),\ \ldots, \\
& 0 + 0 \leq 0,\ 0 + 0 \leq s(0),\ \ldots, \\
& \ldots,\ (s(0) + 0) + s(0) \leq s(0) + (s(0) + s(0)) \\
& \ldots \\
& s(0) + 0 < s(0) + 0 + 0 + s(0) \\
& \ldots\}
\end{aligned}
$$

Existence of Herbrand Models

A Herbrand interpretation I is called a *Herbrand model* of F, if $I \models F$.

Theorem 3.13 (Herbrand) *Let N be a set of Σ-clauses.*

$$
\begin{aligned}
N \text{ satisfiable} \quad & \Leftrightarrow \quad N \text{ has a Herbrand model (over } \Sigma) \\
& \Leftrightarrow \quad G_\Sigma(N) \text{ has a Herbrand model (over } \Sigma)
\end{aligned}
$$

where $G_\Sigma(N) = \{\, C\sigma \text{ ground clause} \mid C \in N,\ \sigma : X \to \mathrm{T}_\Sigma \,\}$ *is the set of* ground instances *of* N.

[The proof will be given below in the context of the completeness proof for resolution.]

Example of a G_Σ

For Σ_{Pres} one obtains for

$$C = (x < y) \vee (y \leq s(x))$$

the following ground instances:

$(0 < 0) \vee (0 \leq s(0))$
$(s(0) < 0) \vee (0 \leq s(s(0)))$
$\ldots$
$(s(0) + s(0) < s(0) + 0) \vee (s(0) + 0 \leq s(s(0) + s(0)))$
$\ldots$