3.3 Models, Validity, and Satisfiability

F is *valid* in $\mathcal{A}$ under assignment β:

$$\mathcal{A}, \beta \models F \quad :\Leftrightarrow \quad \mathcal{A}(\beta)(F) = 1$$

F is *valid* in $\mathcal{A}$ ($\mathcal{A}$ is a *model* of F):

$$\mathcal{A} \models F \quad :\Leftrightarrow \quad \mathcal{A}, \beta \models F, \text{ for all } \beta \in X \to U_{\mathcal{A}}$$

F is *valid* (or is a *tautology*):

$$\models F \quad :\Leftrightarrow \quad \mathcal{A} \models F, \text{ for all } \mathcal{A} \in \Sigma\text{-Alg}$$

F is called *satisfiable* iff there exist $\mathcal{A}$ and β such that $\mathcal{A}, \beta \models F$. Otherwise F is called *unsatisfiable*.

Substitution Lemma

The following propositions, to be proved by structural induction, hold for all Σ-algebras $\mathcal{A}$, assignments β, and substitutions σ.

Lemma 3.3 *For any Σ-term t*

$$\mathcal{A}(\beta)(t\sigma) = \mathcal{A}(\beta \circ \sigma)(t),$$

where $\beta \circ \sigma : X \to \mathcal{A}$ is the assignment $\beta \circ \sigma(x) = \mathcal{A}(\beta)(x\sigma)$.

Proposition 3.4 *For any Σ-formula F, $\mathcal{A}(\beta)(F\sigma) = \mathcal{A}(\beta \circ \sigma)(F)$.*

Corollary 3.5 $\mathcal{A}, \beta \models F\sigma \quad \Leftrightarrow \quad \mathcal{A}, \beta \circ \sigma \models F$

These theorems basically express that the syntactic concept of substitution corresponds to the semantic concept of an assignment.

Entailment and Equivalence

F *entails* (*implies*) G (or G *is a consequence of* F), written $F \models G$, if for all $\mathcal{A} \in \Sigma$-Alg and $\beta \in X \to U_{\mathcal{A}}$, whenever $\mathcal{A}, \beta \models F$, then $\mathcal{A}, \beta \models G$.

F and G are called *equivalent*, written $F \models\!\!\mid G$, if for all $\mathcal{A} \in \Sigma$-Alg und $\beta \in X \to U_{\mathcal{A}}$ we have $\mathcal{A}, \beta \models F \quad \Leftrightarrow \quad \mathcal{A}, \beta \models G$.

Proposition 3.6 *F entails G iff $(F \to G)$ is valid*

Proposition 3.7 *F and G are equivalent iff $(F \leftrightarrow G)$ is valid.*

Extension to sets of formulas N in the "natural way", e. g., $N \models F$

$:\Leftrightarrow$ for all $\mathcal{A} \in \Sigma$-Alg and $\beta \in X \to U_{\mathcal{A}}$: if $\mathcal{A}, \beta \models G$, for all $G \in N$, then $\mathcal{A}, \beta \models F$.

Validity vs. Unsatisfiability

Validity and unsatisfiability are just two sides of the same medal as explained by the following proposition.

Proposition 3.8 *Let F and G be formulas, let N be a set of formulas. Then*

(i) *F is valid if and only if $\neg F$ is unsatisfiable.*

(ii) *$F \models G$ if and only if $F \wedge \neg G$ is unsatisfiable.*

(iii) *$N \models G$ if and only if $N \cup \{\neg G\}$ is unsatisfiable.*

Hence in order to design a theorem prover (validity checker) it is sufficient to design a checker for unsatisfiability.

Theory of a Structure

Let $\mathcal{A} \in \Sigma$-Alg. The *(first-order) theory* of $\mathcal{A}$ is defined as

$$Th(\mathcal{A}) = \{\, G \in \mathrm{F}_\Sigma(X) \mid \mathcal{A} \models G \,\}$$

Problem of axiomatizability:

For which structures $\mathcal{A}$ can one *axiomatize* $Th(\mathcal{A})$, that is, can one write down a formula F (or a recursively enumerable set F of formulas) such that

$$Th(\mathcal{A}) = \{\, G \mid F \models G \,\}?$$

Analogously for sets of structures.

Two Interesting Theories

Let $\Sigma_{Pres} = (\{0/0, s/1, +/2\},\ \emptyset)$ and $\mathbb{Z}_+ = (\mathbb{Z}, 0, s, +)$ its standard interpretation on the integers. $Th(\mathbb{Z}_+)$ is called *Presburger arithmetic* (M. Presburger, 1929). (There is no essential difference when one, instead of $\mathbb{Z}$, considers the natural numbers $\mathbb{N}$ as standard interpretation.)

Presburger arithmetic is decidable in 3EXPTIME (D. Oppen, JCSS, 16(3):323–332, 1978), and in 2EXPSPACE, using automata-theoretic methods (and there is a constant $c \geq 0$ such that $Th(\mathbb{Z}_+) \not\in \text{NTIME}(2^{2^{cn}})$).

However, $\mathbb{N}_* = (\mathbb{N}, 0, s, +, *)$, the standard interpretation of $\Sigma_{PA} = (\{0/0, s/1, +/2, */2\}, \emptyset)$, has as theory the so-called *Peano arithmetic* which is undecidable, not even recursively enumerable.

Note: The choice of signature can make a big difference with regard to the computational complexity of theories.

3.4 Algorithmic Problems

Validity(F): $\models F$?

Satisfiability(F): F satisfiable?

Entailment(F,G): does F entail G?

Model(A,F): $A \models F$?

Solve(A,F): find an assignment β such that $A, \beta \models F$.

Solve(F): find a substitution σ such that $\models F\sigma$.

Abduce(F): find G with “certain properties” such that $G \models F$.

Gödel’s Famous Theorems

1. For most signatures Σ, validity is undecidable for Σ-formulas. (One can easily encode Turing machines in most signatures.)
2. For each signature Σ, the set of valid Σ-formulas is recursively enumerable. (We will prove this by giving complete deduction systems.)
3. For $\Sigma = \Sigma_{PA}$ and $\mathbb{N}_* = (\mathbb{N}, 0, s, +, *)$, the theory $Th(\mathbb{N}_*)$ is not recursively enumerable.

These complexity results motivate the study of subclasses of formulas (*fragments*) of first-order logic

Q: Can you think of any fragments of first-order logic for which validity is decidable?

Some Decidable Fragments

Some decidable fragments:

- *Monadic class*: no function symbols, all predicates unary; validity is NEXPTIME-complete.
- Variable-free formulas without equality: satisfiability is NP-complete. (why?)
- Variable-free Horn clauses (clauses with at most one positive atom): entailment is decidable in linear time.
- Finite model checking is decidable in time polynomial in the size of the structure and the formula.

3.5 Normal Forms and Skolemization

Study of normal forms motivated by

- reduction of logical concepts,
- efficient data structures for theorem proving.

The main problem in first-order logic is the treatment of quantifiers. The subsequent normal form transformations are intended to eliminate many of them.

Prenex Normal Form

Prenex formulas have the form

$$Q_1 x_1 \ldots Q_n x_n \; F,$$

where F is quantifier-free and $Q_i \in \{\forall, \exists\}$; we call $Q_1 x_1 \ldots Q_n x_n$ the *quantifier prefix* and F the *matrix* of the formula.

Computing prenex normal form by the rewrite relation $\Rightarrow_P$:

$$\begin{array}{rcll}
(F \leftrightarrow G) & \Rightarrow_P & (F \rightarrow G) \wedge (G \rightarrow F) & \\
\neg Q x F & \Rightarrow_P & \overline{Q} x \neg F & (\neg Q) \\
((Q x F) \; \rho \; G) & \Rightarrow_P & Q y (F\{x \mapsto y\} \; \rho \; G), \;\; \rho \in \{\wedge, \vee\} & \\
((Q x F) \rightarrow G) & \Rightarrow_P & \overline{Q} y (F\{x \mapsto y\} \rightarrow G), & \\
(F \; \rho \; (Q x G)) & \Rightarrow_P & Q y (F \; \rho \; G\{x \mapsto y\}), \;\; \rho \in \{\wedge, \vee, \rightarrow\} &
\end{array}$$

Here y is always assumed to be some fresh variable and $\overline{Q}$ denotes the quantifier *dual* to Q, i. e., $\overline{\forall} = \exists$ and $\overline{\exists} = \forall$.

Skolemization

Intuition: replacement of $\exists y$ by a concrete choice function computing y from all the arguments y depends on.

Transformation $\Rightarrow_S$ (to be applied outermost, *not* in subformulas):

$$\forall x_1, \ldots, x_n \exists y F \quad \Rightarrow_S \quad \forall x_1, \ldots, x_n F\{y \mapsto f(x_1, \ldots, x_n)\}$$

where f/n is a new function symbol (*Skolem function*).

Together: $F \Rightarrow_P^* \underbrace{G}_{\text{prenex}} \Rightarrow_S^* \underbrace{H}_{\text{prenex, no } \exists}$

Theorem 3.9 *Let F, G, and H as defined above and closed. Then*

(i) F and G are equivalent.

(ii) $H \models G$ but the converse is not true in general.

(iii) G satisfiable (w. r. t. Σ-Alg) $\Leftrightarrow$ H satisfiable (w. r. t. Σ'-Alg) where $\Sigma' = (\Omega \cup SKF, \Pi)$, if $\Sigma = (\Omega, \Pi)$.

Clausal Normal Form (Conjunctive Normal Form)

$$\begin{array}{rcl}
(F \leftrightarrow G) & \Rightarrow_K & (F \rightarrow G) \wedge (G \rightarrow F) \\
(F \rightarrow G) & \Rightarrow_K & (\neg F \vee G) \\
\neg(F \vee G) & \Rightarrow_K & (\neg F \wedge \neg G) \\
\neg(F \wedge G) & \Rightarrow_K & (\neg F \vee \neg G) \\
\neg\neg F & \Rightarrow_K & F \\
(F \wedge G) \vee H & \Rightarrow_K & (F \vee H) \wedge (G \vee H) \\
(F \wedge \top) & \Rightarrow_K & F \\
(F \wedge \bot) & \Rightarrow_K & \bot \\
(F \vee \top) & \Rightarrow_K & \top \\
(F \vee \bot) & \Rightarrow_K & F
\end{array}$$

These rules are to be applied modulo associativity and commutativity of $\wedge$ and $\vee$. The first five rules, plus the rule $(\neg Q)$, compute the *negation normal form* (NNF) of a formula.

The Complete Picture

$$
\begin{array}{lll}
F & \overset{*}{\Rightarrow}_P & Q_1 y_1 \dots Q_n y_n \; G \qquad\qquad\qquad\qquad (G \text{ quantifier-free}) \\
 & \overset{*}{\Rightarrow}_S & \forall x_1, \dots, x_m \; H \qquad\qquad (m \leq n,\ H \text{ quantifier-free}) \\
 & \overset{*}{\Rightarrow}_K & \underbrace{\underbrace{\forall x_1, \dots, x_m}_{\text{leave out}} \; \bigwedge_{i=1}^{k} \; \underbrace{\bigvee_{j=1}^{n_i} L_{ij}}_{\text{clauses } C_i}}_{F'}
\end{array}
$$

$N = \{C_1, \dots, C_k\}$ is called the *clausal (normal) form* (CNF) of F.
Note: the variables in the clauses are implicitly universally quantified.

Theorem 3.10 *Let F be closed. Then $F' \models F$. (The converse is not true in general.)*

Theorem 3.11 *Let F be closed. Then F is satisfiable iff F' is satisfiable iff N is satisfiable*

Optimization

The normal form algorithm described so far leaves lots of room for optimization. Note that we only can preserve satisfiability anyway due to Skolemization.

- size of the CNF is exponential when done naively; the transformations we introduced already for propositional logic avoid this exponential growth;
- we want to preserve the original formula structure;
- we want small arity of Skolem functions (see next section).