3.10 Refutational Completeness of Resolution

How to show refutational completeness of propositional resolution:

- We have to show: $N \models \bot \;\Rightarrow\; N \vdash_{Res} \bot$, or equivalently: If $N \not\vdash_{Res} \bot$, then N has a model.
- Idea: Suppose that we have computed sufficiently many inferences (and not derived $\bot$).
- Now order the clauses in N according to some appropriate ordering, inspect the clauses in ascending order, and construct a series of Herbrand interpretations.
- The limit interpretation can be shown to be a model of N.

Clause Orderings

1. We assume that $\succ$ is any fixed ordering on ground atoms that is *total* and *well-founded*. (There exist many such orderings, e. g., the lenght-based ordering on atoms when these are viewed as words over a suitable alphabet.)
2. Extend $\succ$ to an *ordering* $\succ_L$ *on ground literals:*

$$\begin{array}{lcll} [\neg]A & \succ_L & [\neg]B & \text{, if } A \succ B \\ \neg A & \succ_L & A & \end{array}$$

3. Extend $\succ_L$ to an *ordering* $\succ_C$ *on ground clauses:*
   $\succ_C = (\succ_L)_{\mathrm{mul}}$, the multiset extension of $\succ_L$.

   Notation: $\succ$ also for $\succ_L$ and $\succ_C$.

Example

Suppose $A_5 \succ A_4 \succ A_3 \succ A_2 \succ A_1 \succ A_0$. Then:

$$\begin{array}{lc} & A_5 \vee \neg A_5 \\ \succ & A_3 \vee \neg A_4 \\ \succ & \neg A_1 \vee A_3 \vee A_4 \\ \succ & \neg A_1 \vee A_2 \\ \succ & A_1 \vee A_2 \\ \succ & A_0 \vee A_1 \end{array}$$

Properties of the Clause Ordering

Proposition 3.16

1. *The orderings on literals and clauses are total and well-founded.*
2. *Let C and D be clauses with $A = \max(C)$, $B = \max(D)$, where $\max(C)$ denotes the maximal atom in C.*
   (i) If $A \succ B$ then $C \succ D$.
   (ii) If $A = B$, A occurs negatively in C but only positively in D, then $C \succ D$.

Stratified Structure of Clause Sets

Let $B \succ A$. Clause sets are then stratified in this form:

$$\succ \qquad \begin{array}{r|l|l}
 & & \\ \hline
B \left\{ \begin{array}{l} \\ \\ \\ \end{array} \right. & \begin{array}{l} \neg B \vee \ldots \\ \ldots \vee B \vee B \\ \ldots \vee B \end{array} & \quad \text{all clauses } D \text{ with } \max(D) = B \\ \hline
 & \quad \vdots & \\ \hline
A \left\{ \begin{array}{l} \\ \\ \\ \end{array} \right. & \begin{array}{l} \neg A \vee \ldots \\ \ldots \vee A \vee A \\ \ldots \vee A \end{array} & \quad \text{all clauses } C \text{ with } \max(C) = A \\ \hline
 & &
\end{array}$$

Closure of Clause Sets under Res

$$\begin{aligned}
Res(N) &= \{\, C \mid C \text{ is conclusion of an inference in } Res \\
&\qquad\qquad \text{with premises in } N \,\} \\
Res^0(N) &= N \\
Res^{n+1}(N) &= Res(Res^n(N)) \cup Res^n(N), \ \text{for } n \geq 0 \\
Res^*(N) &= \textstyle\bigcup_{n \geq 0} Res^n(N)
\end{aligned}$$

N is called *saturated* (w. r. t. resolution), if $Res(N) \subseteq N$.

Proposition 3.17

(i) $Res^*(N)$ is saturated.

(ii) Res is refutationally complete, iff for each set N of ground clauses:

$$N \models \bot \ \Leftrightarrow \ \bot \in Res^*(N)$$

Construction of Interpretations

Given: set N of ground clauses, atom ordering $\succ$.
Wanted: Herbrand interpretation I such that

- "many" clauses from N are valid in I;
- $I \models N$, *if* N *is saturated and* $\bot \notin N$.

Construction according to $\succ$, starting with the minimal clause.

Main Ideas of the Construction

- Clauses are considered in the order given by $\prec$.
- When considering C, one already has a partial interpretation I_C (initially $I_C = \emptyset$) available.
- If C is true in the partial interpretation I_C, nothing is done. ($\Delta_C = \emptyset$).
- If C is false, one would like to change I_C such that C becomes true.
- Changes should, however, be *monotone*. One never deletes anything from I_C and the truth value of clauses smaller than C should be maintained the way it was in I_C.
- Hence, one chooses $\Delta_C = \{A\}$ if, and only if, C is false in I_C, if A occurs positively in C (*adding* A *will make* C *become true*) and if this occurrence in C is strictly maximal in the ordering on literals (*changing the truth value of* A *has no effect on smaller clauses*).

Construction of Candidate Interpretations

Let $N, \succ$ be given. We define sets I_C and Δ_C for all ground clauses C over the given signature inductively over $\succ$:

$$I_C := \bigcup_{C \succ D} \Delta_D$$

$$\Delta_C := \begin{cases} \{A\}, & \text{if } C \in N,\ C = C' \vee A,\ A \succ C',\ I_C \not\models C \\ \emptyset, & \text{otherwise} \end{cases}$$

We say that C *produces* A, if $\Delta_C = \{A\}$.

The *candidate interpretation* for N (w. r. t. $\succ$) is given as $I_N^{\succ} := \bigcup_C \Delta_C$. (We also simply write I_N or I for $I_N^{\succ}$ if $\succ$ is either irrelevant or known from the context.)

Example

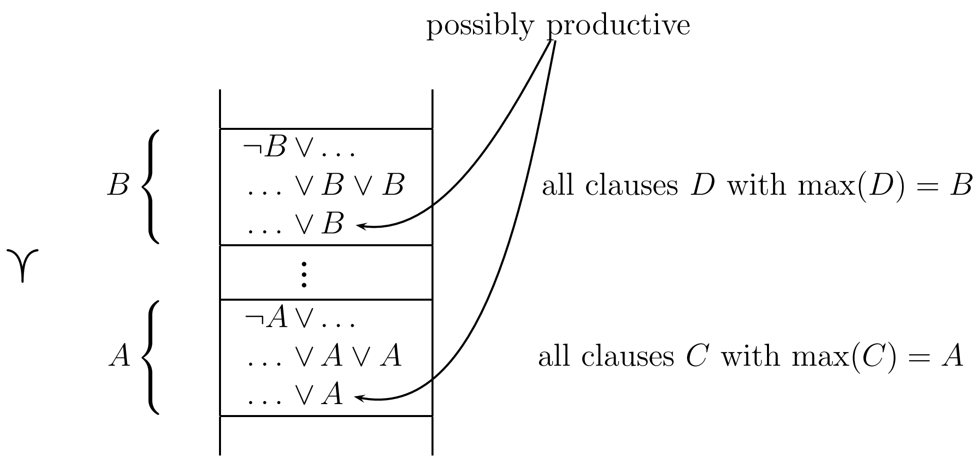
Let $A_5 \succ A_4 \succ A_3 \succ A_2 \succ A_1 \succ A_0$ (max. literals in red)

	clauses C	I_C	Δ_C	Remarks
7	$\neg A_1 \vee A_5$	$\{A_1, A_2, A_4\}$	$\{A_5\}$	
6	$\neg A_1 \vee A_3 \vee \neg A_4$	$\{A_1, A_2, A_4\}$	$\emptyset$	A_3 not maximal; *min. counter-ex.*
5	$A_0 \vee \neg A_1 \vee A_3 \vee A_4$	$\{A_1, A_2\}$	$\{A_4\}$	A_4 maximal
4	$\neg A_1 \vee A_2$	$\{A_1\}$	$\{A_2\}$	A_2 maximal
3	$A_1 \vee A_2$	$\{A_1\}$	$\emptyset$	true in I_C
2	$A_0 \vee A_1$	$\emptyset$	$\{A_1\}$	A_1 maximal
1	$\neg A_0$	$\emptyset$	$\emptyset$	true in I_C

$I = \{A_1, A_2, A_4, A_5\}$ is not a model of the clause set
$\Rightarrow$ there exists a *counterexample*.

Structure of $N, \succ$

Let $B \succ A$; producing a new atom does not affect smaller clauses.



Some Properties of the Construction

Proposition 3.18

(i) $C = \neg A \vee C' \;\Rightarrow\;$ *no* $D \succeq C$ *produces* A.

(ii) C *productive* $\Rightarrow I_C \cup \Delta_C \models C$.

(iii) Let $D' \succ D \succeq C$. *Then*

$$I_D \cup \Delta_D \models C \Rightarrow I_{D'} \cup \Delta_{D'} \models C \text{ and } I_N \models C.$$

If, in addition, $C \in N$ *or* $\max(D) \succ \max(C)$*:*

$$I_D \cup \Delta_D \not\models C \Rightarrow I_{D'} \cup \Delta_{D'} \not\models C \text{ and } I_N \not\models C.$$

(iv) Let $D' \succ D \succ C$*. Then*

$$I_D \models C \Rightarrow I_{D'} \models C \text{ and } I_N \models C.$$

If, in addition, $C \in N$ *or* $\max(D) \succ \max(C)$*:*

$$I_D \not\models C \Rightarrow I_{D'} \not\models C \text{ and } I_N \not\models C.$$

(v) $D = C \vee A$ *produces* $A \Rightarrow I_N \not\models C$*.*

Resolution Reduces Counterexamples

$$\frac{A_0 \vee \neg A_1 \vee A_3 \vee A_4 \qquad \neg A_1 \vee A_3 \vee \neg A_4}{A_0 \vee \neg A_1 \vee \neg A_1 \vee A_3 \vee A_3}$$

Construction of I for the extended clause set:

clauses C	I_C	Δ_C	Remarks
$\neg A_1 \vee A_5$	$\{A_1, A_2, A_4\}$	$\{A_5\}$	
$\neg A_1 \vee A_3 \vee \neg A_4$	$\{A_1, A_2, A_4\}$	$\emptyset$	counterexample
$A_0 \vee \neg A_1 \vee A_3 \vee A_4$	$\{A_1, A_2\}$	$\{A_4\}$	
$A_0 \vee \neg A_1 \vee \neg A_1 \vee A_3 \vee A_3$	$\{A_1, A_2\}$	$\emptyset$	A_3 occurs twice *minimal counter-ex.*
$\neg A_1 \vee A_2$	$\{A_1\}$	$\{A_2\}$	
$A_1 \vee A_2$	$\{A_1\}$	$\emptyset$	
$A_0 \vee A_1$	$\emptyset$	$\{A_1\}$	
$\neg A_0$	$\emptyset$	$\emptyset$	

The same I, but smaller counterexample, hence some progress was made.

Factorization Reduces Counterexamples

$$\frac{A_0 \vee \neg A_1 \vee \neg A_1 \vee A_3 \vee A_3}{A_0 \vee \neg A_1 \vee \neg A_1 \vee A_3}$$

Construction of I for the extended clause set:

clauses C	I_C	Δ_C	Remarks
$\neg A_1 \vee A_5$	$\{A_1, A_2, A_3\}$	$\{A_5\}$	
$\neg A_1 \vee A_3 \vee \neg A_4$	$\{A_1, A_2, A_3\}$	$\emptyset$	true in I_C
$A_0 \vee \neg A_1 \vee A_3 \vee A_4$	$\{A_1, A_2, A_3\}$	$\emptyset$	
$A_0 \vee \neg A_1 \vee \neg A_1 \vee A_3 \vee A_3$	$\{A_1, A_2, A_3\}$	$\emptyset$	true in I_C
$A_0 \vee \neg A_1 \vee \neg A_1 \vee A_3$	$\{A_1, A_2\}$	$\{A_3\}$	
$\neg A_1 \vee A_2$	$\{A_1\}$	$\{A_2\}$	
$A_1 \vee A_2$	$\{A_1\}$	$\emptyset$	
$A_0 \vee A_1$	$\emptyset$	$\{A_1\}$	
$\neg A_0$	$\emptyset$	$\emptyset$	

The resulting $I = \{A_1, A_2, A_3, A_5\}$ is a model of the clause set.

Model Existence Theorem

Theorem 3.19 (Bachmair & Ganzinger 1990) *Let $\succ$ be a clause ordering, let N be saturated w. r. t. Res, and suppose that $\bot \notin N$. Then $I_N^{\succ} \models N$.*

Corollary 3.20 *Let N be saturated w. r. t. Res. Then $N \models \bot \Leftrightarrow \bot \in N$.*

Proof of Theorem 3.19. Suppose $\bot \notin N$, but $I_N^{\succ} \not\models N$. Let $C \in N$ minimal (in $\succ$) such that $I_N^{\succ} \not\models C$. Since C is false in I_N, C is not productive. As $C \neq \bot$ there exists a maximal atom A in C.

Case 1: $C = \neg A \vee C'$ (i. e., the maximal atom occurs negatively)
$\Rightarrow I_N \models A$ and $I_N \not\models C'$
$\Rightarrow$ some $D = D' \vee A \in N$ produces A. Since there is an inference

$$\frac{D' \vee A \qquad \neg A \vee C'}{D' \vee C'},$$

we infer that $D' \vee C' \in N$, and $C \succ D' \vee C'$ and $I_N \not\models D' \vee C'$. This contradicts the minimality of C.

Case 2: $C = C' \vee A \vee A$. There is an inference

$$\frac{C' \vee A \vee A}{C' \vee A}$$

that yields a smaller counterexample $C' \vee A \in N$. This contradicts the minimality of C.

□

Compactness of Propositional Logic

Theorem 3.21 (Compactness) *Let N be a set of propositional formulas. Then N is unsatisfiable, if and only if some finite subset $M \subseteq N$ is unsatisfiable.*

Proof. "$\Leftarrow$": trivial.

"$\Rightarrow$": Let N be unsatisfiable.
$\Rightarrow$ $Res^*(N)$ unsatisfiable
$\Rightarrow$ $\bot \in Res^*(N)$ by refutational completeness of resolution
$\Rightarrow$ $\exists n \geq 0 : \bot \in Res^n(N)$
$\Rightarrow$ $\bot$ has a finite resolution proof P;
choose M as the set of assumptions in P. $\square$

3.11 General Resolution

Propositional resolution:

refutationally complete,

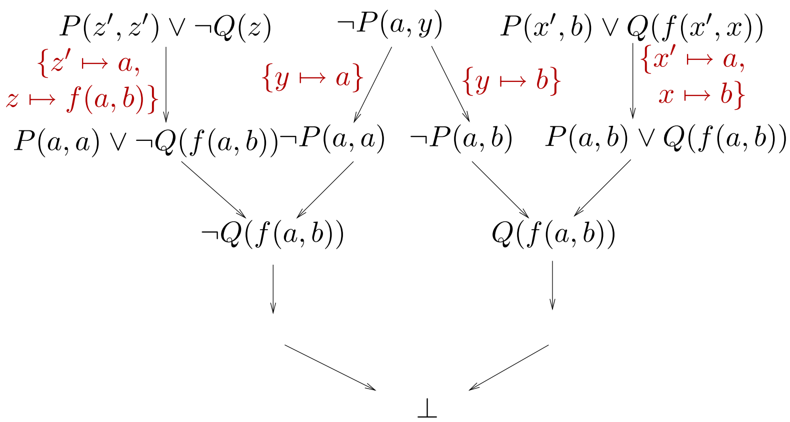
in its most naive version: not guaranteed to terminate for satisfiable sets of clauses, (improved versions do terminate, however)

inferior to the DPLL procedure.

But: in contrast to the DPLL procedure, resolution can be easily extended to non-ground clauses.

General Resolution through Instantiation

Idea: instantiate clauses appropriately:



Problems:

More than one instance of a clause can participate in a proof.

Even worse: There are infinitely many possible instances.

Observation:

Instantiation must produce complementary literals (so that inferences become possible).

Idea:

Do not instantiate more than necessary to get complementary literals.

$$\begin{tikzcd}
P(z',z') \vee \neg Q(z) \arrow[d, "\{z' \mapsto a\}"'] & & & \neg P(a,y) \arrow[dl, "\{y \mapsto a\}"'] \arrow[dr, "\{y \mapsto b\}"] & & & P(x',b) \vee Q(f(x',x)) \arrow[d, "\{x' \mapsto a\}"] \\
P(a,a) \vee \neg Q(z) \arrow[dr] & & \neg P(a,a) \arrow[dl] & & \neg P(a,b) \arrow[dr] & & P(a,b) \vee Q(f(a,x)) \arrow[dl] \\
& \neg Q(z) \arrow[d, "\{z \mapsto f(a,x)\}"'] & & & & Q(f(a,x)) \arrow[d] & \\
& \neg Q(f(a,x)) \arrow[drr] & & & & Q(f(a,x)) \arrow[dll] & \\
& & & \bot & & &
\end{tikzcd}$$

Lifting Principle

Problem: Make saturation of infinite sets of clauses as they arise from taking the (ground) instances of finitely many *general* clauses (with variables) effective and efficient.

Idea (Robinson 1965):

- Resolution for general clauses:
- *Equality* of ground atoms is generalized to *unifiability* of general atoms;
- Only compute *most general* (minimal) unifiers (mgu).

Significance: The advantage of the method in (Robinson 1965) compared with (Gilmore 1960) is that unification enumerates only those instances of clauses that participate in an inference. Moreover, clauses are not right away instantiated into ground clauses. Rather they are instantiated only as far as required for an inference. Inferences with non-ground clauses in general represent infinite sets of ground inferences which are computed simultaneously in a single step.

Resolution for General Clauses

General binary resolution Res:

$$\frac{D \vee B \qquad C \vee \neg A}{(D \vee C)\sigma} \quad \text{if } \sigma = \text{mgu}(A, B) \qquad \text{[resolution]}$$

$$\frac{C \vee A \vee B}{(C \vee A)\sigma} \quad \text{if } \sigma = \text{mgu}(A, B) \quad \text{[factorization]}$$

General resolution RIF with implicit factorization:

$$\frac{D \vee B_1 \vee \ldots \vee B_n \qquad C \vee \neg A}{(D \vee C)\sigma} \quad \text{if } \sigma = \text{mgu}(A, B_1, \ldots, B_n) \qquad \text{[RIF]}$$

For inferences with more than one premise, we assume that the variables in the premises are (bijectively) renamed such that they become different to any variable in the other premises. We do not formalize this. Which names one uses for variables is otherwise irrelevant.

Unification

Let $E = \{s_1 \doteq t_1, \ldots, s_n \doteq t_n\}$ (s_i, t_i terms or atoms) a multiset of *equality problems.* A substitution σ is called a *unifier* of E if $s_i\sigma = t_i\sigma$ for all $1 \leq i \leq n$.

If a unifier of E exists, then E is called *unifiable.*

A substitution σ is called *more general* than a substitution τ, denoted by $\sigma \leq \tau$, if there exists a substitution ρ such that $\rho \circ \sigma = \tau$, where $(\rho \circ \sigma)(x) := (x\sigma)\rho$ is the composition of σ and ρ as mappings. (Note that $\rho \circ \sigma$ has a finite domain as required for a substitution.)

If a unifier of E is more general than any other unifier of E, then we speak of a *most general unifier* of E, denoted by $\text{mgu}(E)$.

Proposition 3.22

(i) $\leq$ *is a quasi-ordering on substitutions, and* $\circ$ *is associative.*

(ii) If $\sigma \leq \tau$ *and* $\tau \leq \sigma$ *(we write* $\sigma \sim \tau$ *in this case), then* $x\sigma$ *and* $x\tau$ *are equal up to (bijective) variable renaming, for any* x *in* X.

A substitution σ is called *idempotent,* if $\sigma \circ \sigma = \sigma$.

Proposition 3.23 *σ is idempotent iff $dom(\sigma) \cap codom(\sigma) = \emptyset$.*

Rule-Based Naive Standard Unification

$$
\begin{array}{rcl}
t \doteq t, E & \Rightarrow_{SU} & E \\
f(s_1, \ldots, s_n) \doteq f(t_1, \ldots, t_n), E & \Rightarrow_{SU} & s_1 \doteq t_1, \ldots, s_n \doteq t_n, E \\
f(\ldots) \doteq g(\ldots), E & \Rightarrow_{SU} & \perp \\
x \doteq t, E & \Rightarrow_{SU} & x \doteq t, E\{t \mapsto x\} \\
 & & \text{if } x \in var(E), x \notin var(t) \\
x \doteq t, E & \Rightarrow_{SU} & \perp \\
 & & \text{if } x \neq t, x \in var(t) \\
t \doteq x, E & \Rightarrow_{SU} & x \doteq t, E \\
 & & \text{if } t \notin X
\end{array}
$$

SU: Main Properties

If $E = x_1 \doteq u_1, \ldots, x_k \doteq u_k$, with x_i pairwise distinct, $x_i \notin var(u_j)$, then E is called an (equational problem in) *solved form* representing the solution $\sigma_E = \{x_1 \mapsto u_1, \ldots, x_k \mapsto u_k\}$.

Proposition 3.24 *If E is a solved form then σ_E is an mgu of E.*

Theorem 3.25

1. *If $E \Rightarrow_{SU} E'$ then σ is a unifier of E iff σ is a unifier of E'*
2. *If $E \Rightarrow^*_{SU} \perp$ then E is not unifiable.*
3. *If $E \Rightarrow^*_{SU} E'$ with E' in solved form, then $\sigma_{E'}$ is an mgu of E.*

Proof. (1) We have to show this for each of the rules. Let's treat the case for the 4th rule here. Suppose σ is a unifier of $x \doteq t$, that is, $x\sigma = t\sigma$. Thus, $\sigma \circ \{x \mapsto t\} = \sigma[x \mapsto t\sigma] = \sigma[x \mapsto x\sigma] = \sigma$. Therefore, for any equation $u \doteq v$ in E: $u\sigma = v\sigma$, iff $u\{x \mapsto t\}\sigma = v\{x \mapsto t\}\sigma$. (2) and (3) follow by induction from (1) using Proposition 3.24. □

Main Unification Theorem

Theorem 3.26 *E is unifiable if and only if there is a most general unifier σ of E, such that σ is idempotent and $dom(\sigma) \cup codom(\sigma) \subseteq var(E)$.*

Proof.

- $\Rightarrow_{SU}$ is Noetherian. A suitable lexicographic ordering on the multisets E (with $\perp$ minimal) shows this. Compare in this order:

1. the number of defined variables (d.h. variables x in equations $x \doteq t$ with $x \notin var(t)$), which also occur outside their definition elsewhere in E;
2. the multiset ordering induced by (i) the size (number of symbols) in an equation; (ii) if sizes are equal consider $x \doteq t$ smaller than $t \doteq x$, if $t \notin X$.

- A system E that is irreducible w. r. t. $\Rightarrow_{SU}$ is either $\bot$ or a solved form.
- Therefore, reducing any E by SU will end (no matter what reduction strategy we apply) in an irreducible E' having the same unifiers as E, and we can read off the mgu (or non-unifiability) of E from E' (Theorem 3.25, Proposition 3.24).
- σ is idempotent because of the substitution in rule 4. $dom(\sigma) \cup codom(\sigma) \subseteq var(E)$, as no new variables are generated.

$\square$

Rule-Based Polynomial Unification

Problem: using $\Rightarrow_{SU}$, an *exponential growth* of terms is possible.

The following unification algorithm avoids this problem, at least if the final solved form is represented as a DAG.

$$
\begin{array}{rcl}
t \doteq t, E & \Rightarrow_{PU} & E \\
f(s_1, \ldots, s_n) \doteq f(t_1, \ldots, t_n), E & \Rightarrow_{PU} & s_1 \doteq t_1, \ldots, s_n \doteq t_n, E \\
f(\ldots) \doteq g(\ldots), E & \Rightarrow_{PU} & \bot \\
x \doteq y, E & \Rightarrow_{PU} & x \doteq y, E\{x \mapsto y\} \\
 & & \text{if } x \in var(E), x \neq y \\
x_1 \doteq t_1, \ldots, x_n \doteq t_n, E & \Rightarrow_{PU} & \bot \\
 & & \text{if there are positions } p_i \text{ with} \\
 & & t_i/p_i = x_{i+1}, t_n/p_n = x_1 \\
 & & \text{and some } p_i \neq \epsilon \\
 & & \\
x \doteq t, E & \Rightarrow_{PU} & \bot \\
 & & \text{if } x \neq t, x \in var(t) \\
t \doteq x, E & \Rightarrow_{PU} & x \doteq t, E \\
 & & \text{if } t \notin X \\
x \doteq t, x \doteq s, E & \Rightarrow_{PU} & x \doteq t, t \doteq s, E \\
 & & \text{if } t, s \notin X \text{ and } |t| \leq |s|
\end{array}
$$

Properties of PU

Theorem 3.27

1. *If $E \Rightarrow_{PU} E'$ then σ is a unifier of E iff σ is a unifier of E'*
2. *If $E \Rightarrow^{*}_{PU} \perp$ then E is not unifiable.*
3. *If $E \Rightarrow^{*}_{PU} E'$ with E' in solved form, then $\sigma_{E'}$ is an mgu of E.*

Note: The solved form of $\Rightarrow_{PU}$ is different form the solved form obtained from $\Rightarrow_{SU}$. In order to obtain the unifier $\sigma_{E'}$, we have to sort the list of equality problems $x_i \doteq t_i$ in such a way that x_i does not occur in t_j for $j < i$, and then we have to compose the substitutions $\{x_1 \mapsto t_1\} \circ \cdots \circ \{x_k \mapsto t_k\}$.

Lifting Lemma

Lemma 3.28 *Let C and D be variable-disjoint clauses. If*

$$\begin{array}{cc} D & C \\ \downarrow \sigma & \downarrow \rho \\ \multicolumn{2}{c}{\dfrac{D\sigma \qquad C\rho}{C'}} \end{array} \qquad \textit{[propositional resolution]}$$

then there exists a substitution τ such that

$$\begin{array}{c} \dfrac{D \qquad C}{C''} \\ \downarrow \tau \\ C' = C''\tau \end{array} \qquad \textit{[general resolution]}$$

An analogous lifting lemma holds for factorization.

Saturation of Sets of General Clauses

Corollary 3.29 *Let N be a set of general clauses saturated under Res, i. e., $Res(N) \subseteq N$. Then also $G_\Sigma(N)$ is saturated, that is,*

$$Res(G_\Sigma(N)) \subseteq G_\Sigma(N).$$

Proof. W.l.o.g. we may assume that clauses in N are pairwise variable-disjoint. (Otherwise make them disjoint, and this renaming process changes neither $Res(N)$ nor $G_\Sigma(N)$.)

Let $C' \in Res(G_\Sigma(N))$, meaning (i) there exist resolvable ground instances $D\sigma$ and $C\rho$ of N with resolvent C', or else (ii) C' is a factor of a ground instance $C\sigma$ of C.

Case (i): By the Lifting Lemma, D and C are resolvable with a resolvent C'' with $C''\tau = C'$, for a suitable substitution τ. As $C'' \in N$ by assumption, we obtain that $C' \in G_\Sigma(N)$.

Case (ii): Similar. $\square$

Herbrand's Theorem

Lemma 3.30 *Let N be a set of Σ-clauses, let $\mathcal{A}$ be an interpretation. Then $\mathcal{A} \models N$ implies $\mathcal{A} \models G_\Sigma(N)$.*

Lemma 3.31 *Let N be a set of Σ-clauses, let $\mathcal{A}$ be a* Herbrand *interpretation. Then $\mathcal{A} \models G_\Sigma(N)$ implies $\mathcal{A} \models N$.*

Theorem 3.32 (Herbrand) *A set N of Σ-clauses is satisfiable if and only if it has a Herbrand model over Σ.*

Proof. The "$\Leftarrow$" part is trivial. For the "$\Rightarrow$" part let $N \not\models \bot$.

$$
\begin{aligned}
N \not\models \bot &\Rightarrow \bot \notin Res^*(N) \qquad \text{(resolution is sound)}\\
&\Rightarrow \bot \notin G_\Sigma(Res^*(N))\\
&\Rightarrow I_{G_\Sigma(Res^*(N))} \models G_\Sigma(Res^*(N)) \qquad \text{(Thm. 3.19; Cor. 3.29)}\\
&\Rightarrow I_{G_\Sigma(Res^*(N))} \models Res^*(N) \qquad \text{(Lemma 3.31)}\\
&\Rightarrow I_{G_\Sigma(Res^*(N))} \models N \qquad (N \subseteq Res^*(N)) \quad \square
\end{aligned}
$$

The Theorem of Löwenheim-Skolem

Theorem 3.33 (Löwenheim–Skolem) *Let Σ be a countable signature and let S be a set of closed Σ-formulas. Then S is satisfiable iff S has a model over a countable universe.*

Proof. If both X and Σ are countable, then S can be at most countably infinite. Now generate, maintaining satisfiability, a set N of clauses from S. This extends Σ by at most countably many new Skolem functions to Σ'. As Σ' is countable, so is $T_{\Sigma'}$, the universe of Herbrand-interpretations over Σ'. Now apply Theorem 3.32. $\square$

Refutational Completeness of General Resolution

Theorem 3.34 *Let N be a set of general clauses where $Res(N) \subseteq N$. Then*

$$N \models \bot \Leftrightarrow \bot \in N.$$

Proof. Let $Res(N) \subseteq N$. By Corollary 3.29: $Res(G_\Sigma(N)) \subseteq G_\Sigma(N)$

$$\begin{aligned} N \models \bot &\Leftrightarrow G_\Sigma(N) \models \bot && \text{(Lemma 3.30/3.31; Theorem 3.32)} \\ &\Leftrightarrow \bot \in G_\Sigma(N) && \text{(propositional resolution sound and complete)} \\ &\Leftrightarrow \bot \in N \quad \Box \end{aligned}$$

Compactness of Predicate Logic

Theorem 3.35 (Compactness Theorem for First-Order Logic) *Let Φ be a set of first-order formulas. Φ is unsatisfiable $\Leftrightarrow$ some finite subset $\Psi \subseteq \Phi$ is unsatisfiable.*

Proof. The "$\Leftarrow$" part is trivial. For the "$\Rightarrow$" part let Φ be unsatisfiable and let N be the set of clauses obtained by Skolemization and CNF transformation of the formulas in Φ. Clearly $Res^*(N)$ is unsatisfiable. By Theorem 3.34, $\bot \in Res^*(N)$, and therefore $\bot \in Res^n(N)$ for some $n \in \mathbb{N}$. Consequently, $\bot$ has a finite resolution proof B of depth $\leq n$. Choose Ψ as the subset of formulas in Φ such that the corresponding clauses contain the assumptions (leaves) of B. $\Box$