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Tutorials for "Automated Reasoning"
Solution to the exercise sheet 1

Exercise 1.1: *(3 P)*
Determine which of the following formulas are valid/satisfiable/unsatisfiable (don't use truth tables):

(1) $(P \land Q) \rightarrow (P \lor Q)$.

Solution.

$$\begin{array}{rcl}
(P \land Q) \rightarrow (P \lor Q) & \models\!\!\!\dashv & \neg(P \land Q) \lor (P \lor Q) \\
& \models\!\!\!\dashv & \neg P \lor \neg Q \lor P \lor Q \\
& \models\!\!\!\dashv & (\neg P \lor P) \lor (\neg Q \lor Q) \\
& \models\!\!\!\dashv & \top \lor \top \\
& \models\!\!\!\dashv & \top.
\end{array}$$

For any Π-valuation $\mathcal{A}$, we have $\mathcal{A}((P \land Q) \rightarrow (P \lor Q)) = \mathcal{A}(\top) = 1$, hence the given formula is valid.

(2) $(P \lor Q) \rightarrow (P \land Q)$.

Solution.

$$\begin{array}{rcl}
(P \lor Q) \rightarrow (P \land Q) & \models\!\!\!\dashv & \neg(P \lor Q) \lor (P \land Q) \\
& \models\!\!\!\dashv & (\neg(P \lor Q) \lor P) \land (\neg(P \lor Q) \lor Q) \\
& \models\!\!\!\dashv & ((\neg P \land \neg Q) \lor P) \land ((\neg P \land \neg Q) \lor Q) \\
& \models\!\!\!\dashv & ((\neg P \lor P) \land (\neg Q \lor P)) \land ((\neg P \lor Q) \land (\neg Q \lor Q)) \\
& \models\!\!\!\dashv & (\top \land (\neg Q \lor P)) \land ((\neg P \lor Q) \land \top) \\
& \models\!\!\!\dashv & (\neg Q \lor P) \land (\neg P \lor Q) \\
& \models\!\!\!\dashv & (Q \rightarrow P) \land (P \rightarrow Q) \\
& \models\!\!\!\dashv & (Q \leftrightarrow P).
\end{array}$$

For any Π-valuation $\mathcal{A}$, under which Q and P have the same value, the formula evaluates to 1, and for other valuations the formula evaluates to 0, hence the given formula is satisfiable, but not valid.

(3) $(\neg P \to Q) \to ((\neg P \to \neg Q) \to P)$

Solution.

$$\begin{aligned}
(\neg P \to Q) \to ((\neg P \to \neg Q) \to P) &\;\models\!\!\!\!\dashv\; \neg(\neg\neg P \vee Q) \vee (\neg(\neg\neg P \vee \neg Q) \vee P) \\
&\;\models\!\!\!\!\dashv\; (\neg P \wedge \neg Q) \vee ((\neg P \wedge Q) \vee P) \\
&\;\models\!\!\!\!\dashv\; (\neg P \wedge \neg Q) \vee (\neg P \wedge Q) \vee P \\
&\;\models\!\!\!\!\dashv\; (\neg P \wedge \neg Q) \vee P \vee (\neg P \wedge Q) \vee P \\
&\;\models\!\!\!\!\dashv\; ((\neg P \vee P) \wedge (\neg Q \vee P)) \vee ((\neg P \vee P) \wedge (Q \vee P)) \\
&\;\models\!\!\!\!\dashv\; (\top \wedge (\neg Q \vee P)) \vee (\top \wedge (Q \vee P)) \\
&\;\models\!\!\!\!\dashv\; \neg Q \vee P \vee Q \vee P \\
&\;\models\!\!\!\!\dashv\; \neg Q \vee Q \vee P \vee P \\
&\;\models\!\!\!\!\dashv\; \top \vee P \\
&\;\models\!\!\!\!\dashv\; \top.
\end{aligned}$$

For any Π-valuation $\mathcal{A}$, we have $\mathcal{A}((\neg P \to Q) \to ((\neg P \to \neg Q) \to P)) = \mathcal{A}(\top) = 1$, hence the given formula is valid.

(4) $\neg(P \to \neg P)$

Solution.

$$\begin{aligned}
\neg(P \to \neg P) &\;\models\!\!\!\!\dashv\; \neg(\neg P \vee \neg P) \\
&\;\models\!\!\!\!\dashv\; \neg(\neg P) \\
&\;\models\!\!\!\!\dashv\; P.
\end{aligned}$$

The obtained formula is the both CNF and DNF of the original formula. Since every conjunct/disjunct of it does not contain complementary literals, the original formula is neither valid nor unsatisfiable, therefore it is satisfiable.

(5) $\neg(P \vee \neg(P \wedge Q))$

Solution.

$$\begin{aligned}
\neg(P \vee \neg(P \wedge Q)) &\;\models\!\!\!\!\dashv\; \neg P \wedge \neg\neg(P \wedge Q) \\
&\;\models\!\!\!\!\dashv\; \neg P \wedge (P \wedge Q) \\
&\;\models\!\!\!\!\dashv\; (\neg P \wedge P) \wedge Q \\
&\;\models\!\!\!\!\dashv\; \bot \wedge Q \\
&\;\models\!\!\!\!\dashv\; \bot.
\end{aligned}$$

For any Π-valuation $\mathcal{A}$, we have $\mathcal{A}(\neg(P \vee \neg(P \wedge Q))) = \mathcal{A}(\bot) = 0$, hence the given formula is unsatisfiable.

(6) $(P \vee \neg Q) \wedge \neg(\neg P \to \neg Q)$

Solution.

$$\begin{aligned}
(P \vee \neg Q) \wedge \neg(\neg P \to \neg Q) &\;\models\!\!\!\!\dashv\; (P \vee \neg Q) \wedge \neg(\neg\neg P \vee \neg Q) \\
&\;\models\!\!\!\!\dashv\; (P \vee \neg Q) \wedge \neg(P \vee \neg Q) \\
&\;\models\; ((R) \wedge \neg(R)) \wedge (R \leftrightarrow (P \vee \neg Q)) \quad \text{(R is a new prop. var.)} \\
&\;\models\!\!\!\!\dashv\; (R \wedge \neg R) \wedge (R \leftrightarrow (P \vee \neg Q)) \\
&\;\models\!\!\!\!\dashv\; \bot \wedge (R \leftrightarrow (P \vee \neg Q)) \\
&\;\models\!\!\!\!\dashv\; \bot.
\end{aligned}$$

For any Π-valuation $\mathcal{A}$, we have $\mathcal{A}\big(((R) \wedge \neg(R)) \wedge (R \leftrightarrow (P \vee \neg Q))\big) = \mathcal{A}(\bot) = 0$. Since we have used only satisfiablity-preserving transformations, the original formula is unsatisfiable.

Exercise 1.2: *(4 P)*
Let F, G be propositional formulas and P be a propositional variable which does not occur in F nor in G. Prove or refute the following propositions:

1. If $F \wedge G$ is valid/satisfiable, then $P \wedge G \wedge (P \to F)$ is valid/satisfiable.

   Solution.

   Assume $F \wedge G$ is satisfiable, meaning that there exists a Π-valuation $\mathcal{A}$, s.t. $\mathcal{A} \models F \wedge G$. Note, that $\mathcal{A} \models F \wedge G \Leftrightarrow \mathcal{A} \models F$ and $\mathcal{A} \models G$.

   Let $\mathcal{A}'$ be a Π-valuation, s.t. $\mathcal{A}'(P) = 1$ and $\mathcal{A}'$ agrees with $\mathcal{A}$ on any other propositional variable, then, since P does not occur in F or G, we have that $\mathcal{A}' \models F$ and $\mathcal{A}' \models G$, therefore $\mathcal{A}'(P \wedge G \wedge (P \to F)) = \mathcal{A}'(P) \wedge \mathcal{A}'(G) \wedge \mathcal{A}'(P \to F) = 1 \wedge 1 \wedge (1 \to 1) = 1$. So, we've found a Π-valuation $\mathcal{A}'$ that models the formula $P \wedge G \wedge (P \to F)$.

   Let $\mathcal{A}''$ be a Π-valuation, s.t. $\mathcal{A}'(P) = 0$, then $\mathcal{A}''(P \wedge G \wedge (P \to F)) = \mathcal{A}''(P) \wedge \mathcal{A}''(G \wedge (P \to F)) = 0 \wedge \mathcal{A}''(G \wedge (P \to F)) = 0$. So, we've found a Π-valuation $\mathcal{A}''$ that does not model the formula $P \wedge G \wedge (P \to F)$.

   Having the Π-valuations $\mathcal{A}'$ and $\mathcal{A}''$, we can conclude that if $F \wedge G$ is valid or satisfiable, then $P \wedge G \wedge (P \to F)$ is not valid but satisfiable.

2. Let G be unsatisfiable and $F \models G$. Then $F \vee G$ satisfiable.

   Solution.

   $F \models G$ iff $\mathcal{A} \models F \to G$, for an arbitrary Π-valuation $\mathcal{A}$. Also, G is unsat., iff $\mathcal{A} \models \neg G$, for an arbitrary Π-valuation $\mathcal{A}$. These two facts give us that for an arbitrary $\mathcal{A}$:

$$\begin{aligned}
\mathcal{A} \models \neg G \text{ and } \mathcal{A} \models F \to G \quad &\Leftrightarrow \quad \mathcal{A}(\neg G) = 1 \text{ and } \mathcal{A}(F \to G) = 1 \\
&\Leftrightarrow \quad \mathcal{A}(\neg G \wedge (F \to G)) = 1 \\
&\Leftrightarrow \quad \mathcal{A}(\neg G \wedge (\neg F \vee G)) = 1 \\
&\Leftrightarrow \quad \mathcal{A}(\neg F \wedge \neg G) = 1 \\
&\Leftrightarrow \quad \mathcal{A}(\neg(F \vee G)) = 1.
\end{aligned}$$

   As the Π-valuation $\mathcal{A}$ was taken arbitrary, we obtain that $\neg(F \vee G)$ is valid and, thus, $(F \vee G)$ is unsatisfiable, or, equivalently, $(F \vee G)$ is not satisfiable.

3. If $F \to G$ is valid, and $G \to H$ is satisfiable, then $F \to H$ is satisfiable.

   Solution.

   We prove the statement by contradiction.

   Assume that $F \to G$ is valid, $G \to H$ is satisfiable, but $F \to H$ is not satisfiable.

   Let $\mathcal{A}$ be an arbitrary Π-valuation.

   Since $F \to H$ is not satisfiable (or, equivalently, it is unsatisfiable), we have that $\mathcal{A}(\neg(F \to H)) = 1$, iff $\mathcal{A}(F \wedge \neg H)) = 1$, iff $\mathcal{A} \models F$ and $\mathcal{A} \models \neg H$. As the $\mathcal{A}$ is taken arbitrary, we conclude that the formulas F and $\neg H$ are valid: $\models F$ and $\models \neg H$.

   Since $F \to G$ is valid, $\mathcal{A}(F \to G) = 1$, iff $\mathcal{A}(\neg F \vee G) = 1$, iff $\mathcal{A}(\neg F) = 1$ or $\mathcal{A}(G) = 1$, iff $\mathcal{A} \models \neg F$ or $\mathcal{A} \models G$, but from what we have already shown, we know that $\mathcal{A} \models F$, hence $\mathcal{A} \models G$. As the $\mathcal{A}$ is taken arbitrary, we conclude that the formula G is valid: $\models G$.

Since $G \to H$ is satisfiable, there exists a Π-valuation $\mathcal{A}'$ s.t. $\mathcal{A}'(G \to H) = 1$, iff $\mathcal{A}'(\neg G) = 1$ or $\mathcal{A}'(H) = 1$, iff $\mathcal{A}' \models \neg G$ or $\mathcal{A}' \models H$, but we have already shown that G is valid, hence $\mathcal{A}' \models H$, but this contradicts the fact that $\neg H$ is valid.

Thus our assumption was wrong and the statement of the exercise holds.

4. If F is satisfiable and G is satisfiable, then $F \wedge G$ is satisfiable.

   Solution.

   We refute the statement by contrexample.

   Let $F = Q$ and $G = \neg Q$, where Q is a propositional variable. F and G are clearly satisfiable, but $F \wedge G$ is not, because $F \wedge G = Q \wedge \neg Q \models\!\!\mid \bot$.

Exercise 1.3: (*2 P*)
Transform the following formula to both CNF and DNF following the conversion steps from the lecture: $((P \to Q) \vee R) \wedge (\neg Q \to P)$.

Solution.

1. CNF.

$$\begin{aligned} ((P \to Q) \vee R) \wedge (\neg Q \to P) \;&\Rightarrow_K^* \; (\neg P \vee Q \vee R) \wedge (\neg\neg Q \vee P) \\ &\Rightarrow_K \; (\neg P \vee Q \vee R) \wedge (Q \vee P). \end{aligned}$$

2. DNF.

$$\begin{aligned} ((P \to Q) \vee R) \quad \wedge \quad & (\neg Q \to P) \\ & \Rightarrow_K^* (\neg P \vee Q \vee R) \wedge (\neg\neg Q \vee P) \\ & \Rightarrow_K (\neg P \vee Q \vee R) \wedge (Q \vee P) \\ & \Rightarrow_K ((\neg P \vee Q) \wedge (Q \vee P)) \vee (R \wedge (Q \vee P)) \\ & \Rightarrow_K^* ((\neg P \wedge (Q \vee P)) \vee (Q \wedge (Q \vee P))) \vee ((R \wedge Q) \vee (R \wedge P)) \\ & \Rightarrow_K^* (\neg P \wedge Q) \vee (\neg P \wedge P) \vee (Q \wedge Q) \vee (Q \wedge P) \vee (R \wedge Q) \vee (R \wedge P). \end{aligned}$$

(We use the notation $\Rightarrow_K^*$ to denote a multiple application of $\Rightarrow_K$.)

Exercise 1.4: (*1 P*)
Let F be a propositional formula. Show how to check its validity using an implementation of the DPLL procedure.

Solution.

A propositional formula F is valid, iff $\neg F$ is unsatisfiable. The DPLL procedure is aimed to check whether a given clause set is satisfiable or not, or, equivalently, the DPLL procedure can be used as an unsatisfiablity checker. Based on these observations, one can check validity of a given formula F in the following way:

1. Compute $F' = \neg F$.
2. Compute $F'' = CNF(F')$, i.e. compute the CNF of F'.
3. If DPLL($\emptyset$, F'') is false, the formula F is valid, otherwise F – not valid.

Challenge Problem: (*2 Bonus Points*)
Let F be a propositional formula which contains no occurrence of $\to$ or $\leftrightarrow$, then F° is the propositional formula obtained by replacing all occurrences of propositional variables by their negations.

The *dual* of F, which we denote here by F^*, is the propositional formula obtained by replacing every occurrence of $\top$ by $\bot$, every occurrence of $\bot$ by $\top$, every occurrence of $\vee$ by $\wedge$ and every occurrence of $\wedge$ by $\vee$.

Prove or refute that $F^* \mathrel{\models\!\!\!\dashv} \neg F^\circ$.

Solution.

We claim that $F^* \mathrel{\models\!\!\!\dashv} \neg F^\circ$ holds.

Proof. We prove the statement by the Principle of Structural Induction.

Basic Step. Suppose F is atomic. Consider possible cases:

- $F = P$, where P is a propositional variable. Then

$$\begin{array}{rcll} F^* & = & P^* & \\ & = & P & \text{(def. of }^*) \\ & \mathrel{\models\!\!\!\dashv} & \neg\neg P & \\ & = & \neg(P^\circ) & \text{(def. of }^\circ) \\ & = & \neg F^\circ. & \end{array}$$

- $F = \top$. Then

$$\begin{array}{rcll} F^* & = & \top^* & \\ & = & \bot & \text{(def. of }^*) \\ & \mathrel{\models\!\!\!\dashv} & \neg\top & \\ & = & \neg(\top^\circ) & \text{(def. of }^\circ) \\ & = & \neg F^\circ. & \end{array}$$

- $F = \bot$. This case is similar to the previous one.

Thus, for every atomic formula F, we have that $F^* \mathrel{\models\!\!\!\dashv} \neg F^\circ$.

Induction Step. Let H and G be arbitrary propositional formulas. Suppose that $H^* \mathrel{\models\!\!\!\dashv} \neg H^\circ$ and $G^* \mathrel{\models\!\!\!\dashv} \neg G^\circ$ (**induction hypothesis**), and $F = H \circ G$, where $\circ \in \{\vee, \wedge\}$. Consider the following cases:

- $F = H \vee G$. Then

$$\begin{array}{rcll} F^* & = & (H \vee G)^* & \\ & = & H^* \wedge G^* & \text{(def. of }^*\text{)} \\ & \mathrel{|}\!\!\equiv\!\!\mathrel{|} & \neg H^\circ \wedge \neg G^\circ & \text{(ind.hypothesis)} \\ & \mathrel{|}\!\!\equiv\!\!\mathrel{|} & \neg(H^\circ \vee G^\circ) & \\ & = & \neg(H \vee G)^\circ & \text{(def. of }^\circ\text{)} \\ & = & \neg F^\circ. & \end{array}$$

- $F = H \wedge G$. This case is similar to the previous one.
- $F = \neg H$. Then

$$\begin{array}{rcll} F^* & = & (\neg H)^* & \\ & = & \neg(H^*) & \text{(def. of }^*\text{)} \\ & \mathrel{|}\!\!\equiv\!\!\mathrel{|} & \neg(\neg H^\circ) & \text{(ind.hypothesis)} \\ & = & \neg(\neg H)^\circ & \text{(def. of }^\circ\text{)} \\ & = & \neg F^\circ. & \end{array}$$

Now it follows by the Principle of Structural Induction that, for every propositinal formula F, the property $F^* \mathrel{|}\!\!\equiv\!\!\mathrel{|} \neg F^\circ$ holds.