Variable Renaming

Rename all variables in F such that there are no two different positions p, q with $F/p = Qx\, G$ and $F/q = Qx\, H$.

Standard Skolemization

Let F be the overall formula, then apply the rewrite rule:

$$\begin{array}{lll} \exists x\, H & \Rightarrow_{SK} & H[f(y_1, \dots, y_n)/x] \\ & & \text{if } F/p = \exists x\, H \text{ and } p \text{ has minimal length,} \\ & & \{y_1, \dots, y_n\} \text{ are the free variables in } \exists x\, H, \\ & & f \text{ is a new function symbol, } \mathsf{arity}(f) = n \end{array}$$

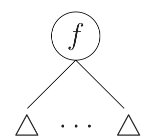
3.7 Herbrand Interpretations

From now an we shall consider PL without equality. Ω shall contains at least one constant symbol.

A *Herbrand interpretation* (over Σ) is a Σ-algebra $\mathcal{A}$ such that

- $U_{\mathcal{A}} = \mathrm{T}_{\Sigma}$ (= the set of ground terms over Σ)
- $f_{\mathcal{A}} : (s_1, \dots, s_n) \mapsto f(s_1, \dots, s_n),\ f \in \Omega, \mathsf{arity}(f) = n$

$f_{\mathcal{A}}(\triangle, \dots, \triangle) =$



In other words, *values are fixed* to be ground terms and *functions are fixed* to be the *term constructors*. Only predicate symbols $P \in \Pi$, $\mathsf{arity}(P) = m$ may be freely interpreted as relations $P_{\mathcal{A}} \subseteq \mathrm{T}_{\Sigma}^{m}$.

Proposition 3.12 *Every set of ground atoms I uniquely determines a Herbrand interpretation $\mathcal{A}$ via*

$$(s_1, \dots, s_n) \in P_{\mathcal{A}} \quad :\Leftrightarrow \quad P(s_1, \dots, s_n) \in I$$

Thus we shall identify Herbrand interpretations (over Σ) with sets of Σ-ground atoms.

Example: $\Sigma_{Pres} = (\{0/0, s/1, +/2\},\ \{</2, \leq/2\})$

$\mathbb{N}$ as Herbrand interpretation over Σ_{Pres}:

$$
\begin{aligned}
I = \{\ \ & 0 \leq 0,\ 0 \leq s(0),\ 0 \leq s(s(0)),\ \ldots, \\
& 0 + 0 \leq 0,\ 0 + 0 \leq s(0),\ \ldots, \\
& \ldots,\ (s(0) + 0) + s(0) \leq s(0) + (s(0) + s(0)) \\
& \ldots \\
& s(0) + 0 < s(0) + 0 + 0 + s(0) \\
& \ldots\}
\end{aligned}
$$

Existence of Herbrand Models

A Herbrand interpretation I is called a *Herbrand model* of F, if $I \models F$.

Theorem 3.13 (Herbrand) *Let N be a set of Σ-clauses.*

$$
\begin{aligned}
N \text{ satisfiable} \quad & \Leftrightarrow \quad N \text{ has a Herbrand model (over } \Sigma) \\
& \Leftrightarrow \quad G_\Sigma(N) \text{ has a Herbrand model (over } \Sigma)
\end{aligned}
$$

where $G_\Sigma(N) = \{C\sigma \text{ ground clause} \mid C \in N,\ \sigma : X \to \mathrm{T}_\Sigma\}$ *is the set of* ground instances *of* N.

[The proof will be given below in the context of the completeness proof for resolution.]

Example of a G_Σ

For Σ_{Pres} one obtains for

$$C = (x < y) \vee (y \leq s(x))$$

the following ground instances:

$(0 < 0) \vee (0 \leq s(0))$
$(s(0) < 0) \vee (0 \leq s(s(0)))$
$\ldots$
$(s(0) + s(0) < s(0) + 0) \vee (s(0) + 0 \leq s(s(0) + s(0)))$
$\ldots$

3.8 Inference Systems and Proofs

Inference systems Γ (proof calculi) are sets of tuples

$$(F_1, \ldots, F_n, F_{n+1}),\ n \geq 0,$$

called *inferences* or *inference rules*, and written

$$\frac{\overbrace{F_1 \ \ldots \ F_n}^{\text{premises}}}{\underbrace{F_{n+1}}_{\text{conclusion}}}\,.$$

Clausal inference system: premises and conclusions are clauses. One also considers inference systems over other data structures (cf. below).

Proofs

A *proof* in Γ of a formula F from a a set of formulas N (called *assumptions*) is a sequence $F_1, \ldots, F_k$ of formulas where

(i) $F_k = F$,

(ii) for all $1 \leq i \leq k$: $F_i \in N$, or else there exists an inference

$$\frac{F_{i_1} \ \ldots \ F_{i_{n_i}}}{F_i}$$

in Γ, such that $0 \leq i_j < i$, for $1 \leq j \leq n_i$.

Soundness and Completeness

Provability $\vdash_\Gamma$ of F from N in Γ: $N \vdash_\Gamma F \ :\Leftrightarrow$ there exists a proof Γ of F from N.

Γ is called *sound* $:\Leftrightarrow$

$$\frac{F_1 \ \ldots \ F_n}{F} \in \Gamma \quad \Rightarrow \quad F_1, \ldots, F_n \models F$$

Γ is called *complete* $:\Leftrightarrow$

$$N \models F \quad \Rightarrow \quad N \vdash_\Gamma F$$

Γ is called *refutationally complete* $:\Leftrightarrow$

$$N \models \bot \quad \Rightarrow \quad N \vdash_\Gamma \bot$$

Proposition 3.14

(i) Let Γ be sound. Then $N \vdash_\Gamma F \Rightarrow N \models F$

(ii) $N \vdash_\Gamma F \Rightarrow$ there exist $F_1, \ldots, F_n \in N$ s.t. $F_1, \ldots, F_n \vdash_\Gamma F$ (resembles compactness).

Proofs as Trees

markings	$\widehat{=}$	formulas		
leaves	$\widehat{=}$	assumptions and axioms		
other nodes	$\widehat{=}$	inferences: conclusion	$\widehat{=}$	ancestor
		premises	$\widehat{=}$	direct descendants

$$\dfrac{P(f(c)) \quad \dfrac{\dfrac{\dfrac{P(f(c)) \lor Q(b) \quad \dfrac{\dfrac{P(f(c)) \lor Q(b) \quad \neg P(f(c)) \lor \neg P(f(c)) \lor Q(b)}{\neg P(f(c)) \lor Q(b) \lor Q(b)}}{\neg P(f(c)) \lor Q(b)}}{Q(b) \lor Q(b)}}{Q(b)} \quad \neg P(f(c)) \lor \neg Q(b)}{\neg P(f(c))}}{\bot}$$

3.9 Propositional Resolution

We observe that propositional clauses and ground clauses are the same concept.

In this section we only deal with ground clauses.

The Resolution Calculus Res

Resolution inference rule:

$$\frac{D \lor A \qquad \neg A \lor C}{D \lor C}$$

Terminology: $D \lor C$: *resolvent*; A: *resolved atom*

(Positive) factorisation inference rule:

$$\frac{C \lor A \lor A}{C \lor A}$$

These are *schematic inference rules*; for each substitution of the *schematic variables* C, D, and A, respectively, by ground clauses and ground atoms we obtain an inference rule.

As "$\lor$" is considered associative and commutative, we assume that A and $\neg A$ can occur anywhere in their respective clauses.

Sample Refutation

1.	$\neg P(f(c)) \vee \neg P(f(c)) \vee Q(b)$	(given)
2.	$P(f(c)) \vee Q(b)$	(given)
3.	$\neg P(g(b,c)) \vee \neg Q(b)$	(given)
4.	$P(g(b,c))$	(given)
5.	$\neg P(f(c)) \vee Q(b) \vee Q(b)$	(Res. 2. into 1.)
6.	$\neg P(f(c)) \vee Q(b)$	(Fact. 5.)
7.	$Q(b) \vee Q(b)$	(Res. 2. into 6.)
8.	$Q(b)$	(Fact. 7.)
9.	$\neg P(g(b,c))$	(Res. 8. into 3.)
10.	$\bot$	(Res. 4. into 9.)

Resolution with Implicit Factorization RIF

$$\frac{D \vee A \vee \ldots \vee A \qquad \neg A \vee C}{D \vee C}$$

1.	$\neg P(f(c)) \vee \neg P(f(c)) \vee Q(b)$	(given)
2.	$P(f(c)) \vee Q(b)$	(given)
3.	$\neg P(g(b,c)) \vee \neg Q(b)$	(given)
4.	$P(g(b,c))$	(given)
5.	$\neg P(f(c)) \vee Q(b) \vee Q(b)$	(Res. 2. into 1.)
6.	$Q(b) \vee Q(b) \vee Q(b)$	(Res. 2. into 5.)
7.	$\neg P(g(b,c))$	(Res. 6. into 3.)
8.	$\bot$	(Res. 4. into 7.)

Soundness of Resolution

Theorem 3.15 *Propositional resolution is sound.*

Proof. Let $I \in \Sigma\text{-}\mathsf{Alg}$. To be shown:

(i) for resolution: $I \models D \vee A,\ I \models C \vee \neg A \ \Rightarrow\ I \models D \vee C$

(ii) for factorization: $I \models C \vee A \vee A \ \Rightarrow\ I \models C \vee A$

(i): Assume premises are valid in I. Two cases need to be considered:
If $I \models A$, then $I \models C$, hence $I \models D \vee C$.
Otherwise, $I \models \neg A$, then $I \models D$, and again $I \models D \vee C$.
(ii): even simpler. □

Note: In propositional logic (ground clauses) we have:

1. $I \models L_1 \vee \ldots \vee L_n \;\Leftrightarrow\;$ there exists i: $I \models L_i$.
2. $I \models A$ or $I \models \neg A$.

This does not hold for formulas with variables!

3.10 Refutational Completeness of Resolution

How to show refutational completeness of propositional resolution:

- We have to show: $N \models \bot \;\Rightarrow\; N \vdash_{Res} \bot$, or equivalently: If $N \not\vdash_{Res} \bot$, then N has a model.
- Idea: Suppose that we have computed sufficiently many inferences (and not derived $\bot$).
- Now order the clauses in N according to some appropriate ordering, inspect the clauses in ascending order, and construct a series of Herbrand interpretations.
- The limit interpretation can be shown to be a model of N.

Clause Orderings

1. We assume that $\succ$ is any fixed ordering on ground atoms that is *total* and *well-founded*. (There exist many such orderings, e. g., the lenght-based ordering on atoms when these are viewed as words over a suitable alphabet.)
2. Extend $\succ$ to an *ordering* $\succ_L$ *on ground literals:*

$$\begin{array}{lll} [\neg]A & \succ_L & [\neg]B \quad \text{, if } A \succ B \\ \neg A & \succ_L & A \end{array}$$

3. Extend $\succ_L$ to an *ordering* $\succ_C$ *on ground clauses:*
   $\succ_C = (\succ_L)_{\text{mul}}$, the multi-set extension of $\succ_L$.

   Notation: $\succ$ also for $\succ_L$ and $\succ_C$.

Example

Suppose $A_5 \succ A_4 \succ A_3 \succ A_2 \succ A_1 \succ A_0$. Then:

$$\begin{array}{cc} & A_0 \vee A_1 \\ \prec & A_1 \vee A_2 \\ \prec & \neg A_1 \vee A_2 \\ \prec & \neg A_1 \vee A_4 \vee A_3 \\ \prec & \neg A_1 \vee \neg A_4 \vee A_3 \\ \prec & \neg A_5 \vee A_5 \end{array}$$