Many-Sorted Structures

A Σ_Υ-*algebra* is a quadruple

$$\begin{aligned}\mathcal{A} = \ & (U_\mathcal{A},\ (f_\mathcal{A} : (T_1)_\mathcal{A} \times \ldots \times (T_n)_\mathcal{A} \to S_\mathcal{A})_{f\in\Omega},\\ & \quad (p_\mathcal{A} \subseteq (S_1)_\mathcal{A} \times \ldots \times (S_m)_\mathcal{A})_{p\in\Pi},\\ & \quad (T_\mathcal{A} \subseteq U_\mathcal{A})_{T\in\Upsilon})\end{aligned}$$

where $\mathsf{arity}(f) = n$, $\mathsf{arity}(p) = m$, $\upsilon(f) = T_1 \ldots T_n S$, $\upsilon(p) = S_1 \ldots S_m$, $T_\mathcal{A} \neq \emptyset$, $U_\mathcal{A} \neq \emptyset$ is a set, called the *universe* of $\mathcal{A}$.

The rest of the semantics is identical to the unsorted case, except that valuations respect the sort information.

7 SUP(LA)

Superposition Modulo Linear Arithmetic

- Consider the base specification $\mathrm{SP} = (\Sigma_{\mathrm{LA}}, \mathcal{A}_{\mathrm{LA}})$, where $\Sigma_{\mathrm{LA}} = (\mathbb{Q} \cup \{+, -, *\}, \{\geq, \leq, >, <\})$ see Section 2.
- The hierarchic extension of SP is $\mathrm{SP}' = (\Sigma', N')$, where $\Sigma_{\mathrm{LA}} \subseteq \Sigma'$ and N' is a set of Σ' clauses.
- We consider a many-sorted setting, consisting of a base sort, containing all terms of Σ_{LA} plus potentially extension terms from $\Sigma' \setminus \Sigma_{\mathrm{LA}}$, and a general sort containing all other terms.
- A term (a clause) consisting only of Σ_{LA} symbols and base sort variables, is called a *base term* (*base clause*).
- For the following results, we need that $\mathcal{A}_{\mathrm{LA}}$ is *term-generated*, i.e., for any $a \in U_{\mathrm{LA}}$ $(= \mathbb{Q})$ there is a ground term $t \in \mathrm{T}_{\Sigma_{\mathrm{LA}}}$ with $\mathcal{A}_{\mathrm{LA}}(t) = a$. This is obvious, because $\mathbb{Q} \subseteq \Sigma_{\mathrm{LA}}$.
- Furthermore, we need that $\mathrm{SP} = (\Sigma_{\mathrm{LA}}, \mathcal{A}_{\mathrm{LA}})$ is compact.
- A model of $\mathcal{A}'$ of SP', i.e., $\mathcal{A}' \models N'$, is called *hierarchic* if $\mathcal{A}' |_{\Sigma_{\mathrm{LA}}} = \mathcal{A}_{\mathrm{LA}}$.
- A substitution is called *simple* if it maps variables of the base sort to base terms.

Hierarchic Clauses

A clause $C = \Lambda \parallel C'$ is called *hierarchic* if Λ only contains base terms and base literals (Σ_{LA}) and all base terms in C' are variables. The semantics of C is $\bigwedge \Lambda \rightarrow C'$.

Any clause can be equivalently transformed into a hierarchic clause: whenever a subterm t whose top symbol is a base theory symbol occurs immediately below a non-base operator symbol, it is replaced by a new base sort variable x ("abstracted out") and the equation $x \approx t$ is added to Λ. Analogously, if a subterm t whose top symbol is not a base theory symbol occurs immediately below a base operator symbol, it is replaced by a general variable y and the disequation $y \not\approx t$ is added to C'. This transformation is repeated until the clause is hierarchic.

Superposition Modulo LA

Pos. Superposition:
$$\frac{\Lambda_1 \parallel D' \vee t \approx t' \qquad \Lambda_2 \parallel C' \vee s[u] \approx s'}{(\Lambda_1, \Lambda_2 \parallel D' \vee C' \vee s[t'] \approx s')\sigma}$$
where $\sigma = \text{mgu}(t, u)$ and simple and u is not a variable.

Neg. Superposition:
$$\frac{\Lambda_1 \parallel D' \vee t \approx t' \qquad \Lambda_2 \parallel C' \vee s[u] \not\approx s'}{(\Lambda_1, \Lambda_2 \parallel D' \vee C' \vee s[t'] \not\approx s')\sigma}$$
where $\sigma = \text{mgu}(t, u)$ and simple and u is not a variable.

Equality Resolution:
$$\frac{\Lambda \parallel C' \vee s \not\approx s'}{(\Lambda \parallel C')\sigma}$$
where $\sigma = \text{mgu}(s, s')$ and simple.

Equality Factoring:
$$\frac{\Lambda \parallel C' \vee s' \approx t' \vee s \approx t}{(\Lambda \parallel C' \vee t \not\approx t' \vee s \approx t')\sigma}$$
where $\sigma = \text{mgu}(s, s')$ and simple.

Constraint Refutation:
$$\frac{\Lambda_1 \parallel \Box \qquad \ldots \qquad \Lambda_n \parallel \Box}{\Box}$$
where $\neg(\bigwedge \Lambda_1) \wedge \ldots \wedge \neg(\bigwedge \Lambda_n)$ is inconsistent in $\mathcal{A}_{\text{LA}}$.

Redundancy

A clause $C \in N$ is called *redundant* if for all simple ground instances C' of C there are simple ground instances $C'_1, \dots, C'_n$ from N such that $C'_1, \dots, C'_n \models C'$ and $C'_i \prec C'$ for all i.

A hierarchic clause $\Lambda \parallel C$ is called a *tautology* if C is a tautology or the existential closure of $\bigwedge \Lambda$ is unsatisfiable in $\mathcal{A}_{\mathrm{LA}}$.

A hierarchic clause $\Lambda_1 \parallel C_1$ *subsumes* a hierarchic clause $\Lambda_2 \parallel C_2$, if there is a simple matcher σ such that $C_1\sigma \subset C_2$ and the universal closure of $\bigwedge \Lambda_2 \rightarrow \bigwedge \Lambda_1\sigma$ holds in $\mathcal{A}_{\mathrm{LA}}$.

Sufficient Completeness

A set N of clauses is called *sufficiently complete with respect to simple instances*, if for every model $\mathcal{A}'$ of the set of simple ground instances from N and every ground non-base term t of the base sort there exists a ground base term t such that $t' \approx t$ is true in $\mathcal{A}'$.

Completeness of SUP(LA)

The hierarchic superposition calculus modulo LA is refutationally complete for all sets of clauses that are sufficiently complete with respect to simple instances.

The End