Case 3.2: $s\theta \not\succ s\theta'$.

If $s\theta \downarrow_{R_C} s\theta'$ and $s\theta \succ s\theta'$, then $s\theta$ must be reducible by some rule in some $E_{D\theta} \subseteq R_C$. (Without loss of generality we assume that $C$ and $D$ are variable disjoint; so we can use the same substitution $\theta$.) Let $D\theta = D'\theta \lor t\theta \approx t'\theta$ with $E_{D\theta} = \{ t\theta \rightarrow t'\theta \}$. Since $D\theta$ is productive, $D'\theta$ is false in $R_C$. Besides, by part (ii) of the induction hypothesis, $D\theta$ is not redundant w.r.t. $G_\Sigma(N)$, so $D$ is not redundant w.r.t. $N$. Note that $t\theta$ cannot occur in $s\theta$ at or below a variable position of $s$, say $x\theta = w[t\theta]$, since otherwise $C\theta$ would be subject to Case 2 above. Consequently, the left superposition inference

\[
\frac{D'\theta \lor t\theta \approx t'\theta \quad C'\theta \lor s\theta[t\theta] \not\approx s'\theta}{D'\theta \lor C'\theta \lor s\theta[t\theta] \not\approx s'\theta}
\]

is a ground instance of a left superposition inference from $D$ and $C$. By saturation up to redundancy, its conclusion is either contained in $G_\Sigma(N)$ and smaller than $C\theta$, or it follows from clauses in $G_\Sigma(N)$ that are smaller than itself (and therefore smaller than $C\theta$). By the induction hypothesis, these clauses are true in $R_C$, thus $D'\theta \lor C'\theta \lor s\theta[t\theta] \not\approx s'\theta$ is true in $R_C$. Since $D'\theta$ and $s\theta[t\theta] \not\approx s'\theta$ are false in $R_C$, both $C'\theta$ and $C\theta$ must be true.

**Case 4: $C\theta$ does not contain a maximal negative literal.**

Suppose that $C\theta$ does not fall into Cases 1 to 3. Then $C\theta$ can be written as $C'\theta \lor s\theta \approx s'\theta$, where $s\theta \approx s'\theta$ is a maximal literal of $C\theta$. If $E_{C\theta} = \{ s\theta \rightarrow s'\theta \}$ or $C'\theta$ is true in $R_C$, or $s\theta = s'\theta$, then there is nothing to show, so assume that $E_{C\theta} = \emptyset$ and that $C'\theta$ is false in $R_C$. Without loss of generality, $s\theta \succ s'\theta$.

**Case 4.1: $s\theta \approx s'\theta$ is maximal in $C\theta$, but not strictly maximal.**

If $s\theta \approx s'\theta$ is maximal in $C\theta$, but not strictly maximal, then $C\theta$ can be written as $C''\theta \lor t\theta \approx t'\theta \lor s\theta \approx s'\theta$, where $t\theta = s\theta$ and $t'\theta = s'\theta$. In this case, there is an equality factoring inference

\[
\frac{C''\theta \lor t\theta \approx t'\theta \lor s\theta \approx s'\theta}{C''\theta \lor t\theta \not\approx s'\theta \lor t\theta \approx t'\theta}
\]

This inference is a ground instance of an inference from $C$. By induction hypothesis, its conclusion is true in $R_C$. Trivially, $t'\theta = s'\theta$ implies $t\theta \downarrow_{R_C} s'\theta$, so $t\theta \not\approx s'\theta$ must be false and $C\theta$ must be true in $R_C$.

**Case 4.2: $s\theta \approx s'\theta$ is strictly maximal in $C\theta$ and $s\theta$ is reducible.**

Suppose that $s\theta \approx s'\theta$ is strictly maximal in $C\theta$ and $s\theta$ is reducible by some rule in $E_{D\theta} \subseteq R_C$. Let $D\theta = D'\theta \lor t\theta \approx t'\theta$ and $E_{D\theta} = \{ t\theta \rightarrow t'\theta \}$. Since $D\theta$ is productive, $D\theta$ is not redundant and $D'\theta$ is false in $R_C$. We can now proceed in essentially the
Case 4.3: If \( t \theta \) occurred in \( s \theta \) at or below a variable position of \( s \), say 
\( x \theta = w[\theta] \), then \( C \theta \) would be subject to Case 2 above. Otherwise, the right superposition inference

\[
\frac{D' \theta \lor t \theta \approx t' \theta}{D' \theta \lor C' \theta \lor s \theta[t' \theta] \approx s' \theta}
\]

is a ground instance of a right superposition inference from \( D \) and \( C \). By saturation up to redundancy, its conclusion is true in \( R_{C \theta} \). Since \( D' \theta \) and \( C' \theta \) are false in \( R_{C \theta} \), \( s' \theta \approx s' \theta \) must be true in \( R_{C \theta} \). On the other hand, \( t \theta \approx t' \theta \) is true in \( R_{C \theta} \), so by congruence, \( s \theta[t\theta] \approx s' \theta \) and \( C \theta \) are true in \( R_{C \theta} \).

Case 4.3: \( s \theta \approx s' \theta \) is strictly maximal in \( C \theta \) and \( s \theta \) is irreducible.

Suppose that \( s \theta \approx s' \theta \) is strictly maximal in \( C \theta \) and \( s \theta \) is irreducible by \( R_{C \theta} \). Then there are three possibilities: \( C \theta \) can be true in \( R_{C \theta} \), or \( C' \theta \) can be true in \( R_{C \theta} \cup \{ s \theta \rightarrow s' \theta \} \), or \( E_{C \theta} = \{ s \theta \rightarrow s' \theta \} \). In the first and the third case, there is nothing to show. Let us therefore assume that \( C \theta \) is false in \( R_{C \theta} \) and \( C' \theta \) is true in \( R_{C \theta} \cup \{ s \theta \rightarrow s' \theta \} \). Then \( C' \theta = C'' \theta \lor t \theta \approx t' \theta \), where the literal \( t \theta \approx t' \theta \) is true in \( R_{C \theta} \cup \{ s \theta \rightarrow s' \theta \} \) and false in \( R_{C \theta} \). In other words, \( t \theta \downarrow_{R_{C \theta} \cup \{ s \theta \rightarrow s' \theta \}} t' \theta \), but not \( t \theta \downarrow_{R_{C \theta}} t' \theta \). Consequently, there is a rewrite proof of \( t \theta \rightarrow^* u \leftarrow^* t' \theta \) by \( R_{C \theta} \cup \{ s \theta \rightarrow s' \theta \} \) in which the rule \( s \theta \rightarrow s' \theta \) is used at least once. Without loss of generality we assume that \( t \theta \succeq t' \theta \). Since \( s \theta \approx s' \theta \succeq_{l} t \theta \approx t' \theta \) and \( s \theta \succeq s' \theta \) we can conclude that \( s \theta \succeq t \theta \succeq t' \theta \). But then there is only one possibility how the rule \( s \theta \rightarrow s' \theta \) can be used in the rewrite proof: We must have \( s \theta = t \theta \) and the rewrite proof must have the form \( t \theta \rightarrow s' \theta \rightarrow^* u \leftarrow^* t' \theta \), where the first step uses \( s \theta \rightarrow s' \theta \) and all other steps use rules from \( R_{C \theta} \). Consequently, \( s' \theta \approx t' \theta \) is true in \( R_{C \theta} \). Now observe that there is an equality factoring inference

\[
\frac{C'' \theta \lor t \theta \approx t' \theta \lor s \theta \approx s' \theta}{C'' \theta \lor t' \theta \not\approx s' \theta \lor t \theta \approx t' \theta}
\]

whose conclusion is true in \( R_{C \theta} \) by saturation. Since the literal \( t' \theta \not\approx s' \theta \) must be false in \( R_{C \theta} \), the rest of the clause must be true in \( R_{C \theta} \), and therefore \( C \theta \) must be true in \( R_{C \theta} \), contradicting our assumption. This concludes the proof of the theorem. \( \square \)

A \( \Sigma \)-interpretation \( A \) is called term-generated, if for every \( b \in U_A \) there is a ground term \( t \in T_{\Sigma}(\emptyset) \) such that \( b = A(\beta)(t) \).

**Lemma 4.53** Let \( N \) be a set of (universally quantified) \( \Sigma \)-clauses and let \( A \) be a term-generated \( \Sigma \)-interpretation. Then \( A \) is a model of \( G_{\Sigma}(N) \) if and only if it is a model of \( N \).
Proof. ($\Rightarrow$): Let $\mathcal{A} \models G_{\Sigma}(N)$; let $(\forall \overline{x}C) \in N$. Then $\mathcal{A} \models \forall \overline{x}C$ iff $\mathcal{A}(\gamma[x_i \mapsto a_i])(C) = 1$ for all $\gamma$ and $a_i$. Choose ground terms $t_i$ such that $\mathcal{A}(\gamma)(t_i) = a_i$. Then $\mathcal{A}(\gamma)(t_i)$ is a model of $N$.

(⇐): Let $\mathcal{A}$ be a model of $N$. Then $\mathcal{A}(\gamma)(C \theta) = 1$ for all $\gamma$ and $a_i$. Choose ground terms $t_i$ such that $\mathcal{A}(\gamma)(C \theta) = 1$ since $C \theta \in G_{\Sigma}(N)$.

Theorem 4.54 (Refutational Completeness: Static View) Let $N$ be a set of clauses that is saturated up to redundancy. Then $N$ has a model if and only if $N$ does not contain the empty clause.

Proof. If $\bot \in N$, then obviously $N$ does not have a model. If $\bot \notin N$, then the interpretation $R_{\infty}$ (that is, $T_{\Sigma}(\emptyset)/R_{\infty}$) is a model of all ground instances in $G_{\Sigma}(N)$ according to part (iii) of the model construction theorem. As $T_{\Sigma}(\emptyset)/R_{\infty}$ is term generated, it is a model of $N$.

So far, we have considered only inference rules that add new clauses to the current set of clauses (corresponding to the Deduce rule of Knuth-Bendix Completion).

In other words, we have derivations of the form $N_0 \vdash N_1 \vdash N_2 \vdots$, where each $N_{i+1}$ is obtained from $N_i$ by adding the consequence of some inference from clauses in $N_i$.

Under which circumstances are we allowed to delete (or simplify) a clause during the derivation?

A run of the superposition calculus is a sequence $N_0 \vdash N_1 \vdash N_2 \vdots$, such that

(i) $N_i \models N_{i+1}$, and

(ii) all clauses in $N_i \setminus N_{i+1}$ are redundant w. r. t. $N_{i+1}$.

In other words, during a run we may add a new clause if it follows from the old ones, and we may delete a clause, if it is redundant w. r. t. the remaining ones.

For a run, $N_\infty = \bigcup_{i \geq 0} N_i$ and $N_* = \bigcap_{i \geq 0} \bigcap_{j \geq i} N_j$. The set $N_*$ of all persistent clauses is called the limit of the run.

Lemma 4.55 If $N \subseteq N'$, then $\text{Red}(N) \subseteq \text{Red}(N')$.

Proof. Obvious. \hfill \Box

Lemma 4.56 If $N' \subseteq \text{Red}(N)$, then $\text{Red}(N) \subseteq \text{Red}(N \setminus N')$.

Proof. Follows from the compactness of first-order logic and the well-foundedness of the multiset extension of the clause ordering. \hfill \Box