4.7 Superposition

Goal:

Combine the ideas of ordered resolution (overlap maximal literals in a clause) and Knuth-Bendix completion (overlap maximal sides of equations) to get a calculus for equational clauses.

Recapitulation: Equational Clauses

Atom: either $p(s_1, \ldots, s_m)$ with $p \in \Pi$ or $s \approx t$.

Literal: Atom or negated atom.

Clause: (possibly empty) disjunction of literals (all variables implicitly universally quantified).

For refutational theorem proving, it is sufficient to consider sets of clauses: every first-order formula F can be translated into a set of clauses N such that F is unsatisfiable if and only if N is unsatisfiable.

In the non-equational case, unsatisfiability can for instance be checked using the (ordered) resolution calculus.

Recapitulation: Ordered Resolution

(Ordered) resolution: inference rules:

	Ground case:	Non-ground case:
Resolution:	$\dfrac{D' \lor A \qquad C' \lor \neg A}{D' \lor C'}$	$\dfrac{D' \lor A \qquad C' \lor \neg A'}{(D' \lor C')\sigma}$ where $\sigma = \mathrm{mgu}(A, A')$.
Factoring:	$\dfrac{C' \lor A \lor A}{C' \lor A}$	$\dfrac{C' \lor A \lor A'}{(C' \lor A)\sigma}$ where $\sigma = \mathrm{mgu}(A, A')$.

Ordering restrictions:

Let $\succ$ be a well-founded and total ordering on ground atoms.

Literal ordering $\succ_L$: compares literals by comparing lexicographically first the respective atoms using $\succ$ and then their polarities (negative $>$ positive).

Clause ordering $\succ_C$: compares clauses by comparing their multisets of literals using the multiset extension of $\succ_L$.

Ordering restrictions (ground case):

Inference are necessary only if the following conditions are satisfied:

- The left premise of a Resolution inference is not larger than or equal to the right premise.
- The literals that are involved in the inferences ($[\neg]\, A$) are maximal in the respective clauses (strictly maximal for the left premise of Resolution).

Ordering restrictions (non-ground case):

Lift the ground ordering to non-ground literals: A literal L is called [strictly] maximal in a clause C if and only if there exists a ground substitution σ such that for all other literals L' in C: $L\sigma \not\prec L'\sigma$ $[L\sigma \not\preceq L'\sigma]$.

Recapitulation: Refutational Completeness

Resolution is (even with ordering restrictions) refutationally complete:

Dynamic view of refutational completeness:

If N is unsatisfiable ($N \models \bot$) then *fair* derivations from N produce $\bot$.

Static view of refutational completeness:

If N is *saturated*, then N is unsatisfiable if and only if $\bot \in N$.

Proving refutational completeness for the ground case:

We have to show:

If N is saturated (i. e., if sufficiently many inferences have been computed), and $\bot \notin N$, then N is satisfiable (i. e., has a model).

Model construction:

Suppose that N be saturated and $\bot \notin N$. We inspect all clauses in N in ascending order and construct a sequence of Herbrand interpretations (starting with the empty interpretation: all atoms are false).

If a clause C is false in the current interpretation, and has a positive and strictly maximal literal A, then extend the current interpretation such that C becomes true: add A to the current interpretation. (Then C is called *productive*.)

Otherwise, leave the current interpretation unchanged.

The sequence of interpretations has the following properties:

(1) If an atom is true in some interpretation, then it remains true in all future interpretations.

(2) If a clause is true at the time where it is inspected, then it remains true in all future interpretations.

(3) If a clause $C = C' \vee A$ is productive, then C remains true and C' remains false in all future interpretations.

Show by induction: if N is saturated and $\bot \notin N$, then every clause in N is either true at the time where it is inspected or productive.

Note:
For the induction proof, it is not necessary that the conclusion of an inference is contained in N. It is sufficient that it is redundant w. r. t. N.

N is called *saturated up to redundancy* if the conclusion of every inference from clauses in $N \setminus \mathit{Red}(N)$ is contained in $N \cup \mathit{Red}(N)$.

Proving refutational completeness for the non-ground case:

If $C_i\theta$ is a ground instance of the clause C_i for $i \in \{0, \dots, n\}$ and

$$\frac{C_n, \dots, C_1}{C_0}$$

and

$$\frac{C_n\theta, \dots, C_1\theta}{C_0\theta}$$

are inferences, then the latter inference is called a *ground instance* of the former.

For a set N of clauses, let $G_\Sigma(N)$ be the set of all ground instances of clauses in N.

Construct the interpretation from the set $G_\Sigma(N)$ of all ground instances of clauses in N:

N is saturated and does not contain $\bot$
$\Rightarrow$ $G_\Sigma(N)$ is saturated and does not contain $\bot$
$\Rightarrow$ $G_\Sigma(N)$ has a Herbrand model I
$\Rightarrow$ I is a model of N.

Observation

It is possible to encode an arbitrary predicate p using a function f_p and a new constant tt:

$$\begin{array}{rcr} p(t_1,\dots,t_n) & \rightsquigarrow & f_p(t_1,\dots,t_n) \approx tt \\ \neg\, p(t_1,\dots,t_n) & \rightsquigarrow & \neg\, f_p(t_1,\dots,t_n) \approx tt \end{array}$$

In equational logic it is therefore sufficient to consider the case that $\Pi = \emptyset$, i. e., equality is the only predicate symbol.

Abbreviation: $s \not\approx t$ instead of $\neg\, s \approx t$.

The Superposition Calculus – Informally

Conventions:

From now on: $\Pi = \emptyset$ (equality is the only predicate).

Inference rules are to be read modulo symmetry of the equality symbol.

We will first explain the ideas and motivations behind the superposition calculus and its completeness proof. Precise definitions will be given later.

Ground inference rules:

Pos. Superposition: $$\frac{D' \lor t \approx t' \qquad C' \lor s[t] \approx s'}{D' \lor C' \lor s[t'] \approx s'}$$

Neg. Superposition: $$\frac{D' \lor t \approx t' \qquad C' \lor s[t] \not\approx s'}{D' \lor C' \lor s[t'] \not\approx s'}$$

Equality Resolution: $$\frac{C' \lor s \not\approx s}{C'}$$

(Note: We will need one further inference rule.)

Ordering restrictions:

Some considerations:

The literal ordering must depend primarily on the larger term of an equation.

As in the resolution case, negative literals must be a bit larger than the corresponding positive literals.

Additionally, we need the following property: If $s \succ t \succ u$, then $s \not\approx u$ must be larger than $s \approx t$. In other words, we must compare first the larger term, then the polarity, and finally the smaller term.

The following construction has the required properties:

Let $\succ$ be a *reduction ordering that is total on ground terms.*

To a positive literal $s \approx t$, we assign the multiset $\{s, t\}$, to a negative literal $s \not\approx t$ the multiset $\{s, s, t, t\}$. The *literal ordering* $\succ_L$ compares these multisets using the multiset extension of $\succ$.

The *clause ordering* $\succ_C$ compares clauses by comparing their multisets of literals using the multiset extension of $\succ_L$.

Ordering restrictions:

Ground inferences are necessary only if the following conditions are satisfied:

- In superposition inferences, the left premise is smaller than the right premise.
- The literals that are involved in the inferences are maximal in the respective clauses (strictly maximal for positive literals in superposition inferences).
- In these literals, the lhs is greater than or equal to the rhs (in superposition inferences: greater than the rhs).

Model construction:

We want to use roughly the same ideas as in the completeness proof for resolution.

But: a Herbrand interpretation does not work for equality: The equality symbol $\approx$ must be interpreted by equality in the interpretation.

Solution: Define a set E of ground equations and take $\mathrm{T}_\Sigma(\emptyset)/E = \mathrm{T}_\Sigma(\emptyset)/{\approx_E}$ as the universe.

Then two ground terms s and t are equal in the interpretation, if and only if $s \approx_E t$.

If E is a terminating and confluent rewrite system R, then two ground terms s and t are equal in the interpretation, if and only if $s \downarrow_R t$.

One problem:

In the completeness proof for the resolution calculus, the following property holds:

If $C = C' \vee A$ with a strictly maximal and positive literal A is false in the current interpretation, then adding A to the current interpretation cannot make any literal of C' true.

This does not hold for superposition:

Let $b \succ c \succ d$. Assume that the current rewrite system (representing the current interpretation) contains the rule $c \to d$. Now consider the clause $b \approx c \vee b \approx d$.

We need a further inference rule to deal with clauses of this kind, either the "Merging Paramodulation" rule of Bachmair and Ganzinger or the following "Equality Factoring" rule due to Nieuwenhuis:

Equality Factoring: $$\frac{C' \vee s \approx t' \vee s \approx t}{C' \vee t \not\approx t' \vee s \approx t'}$$

Note: This inference rule subsumes the usual factoring rule.

How do the non-ground versions of the inference rules for superposition look like?

Main idea as in the resolution calculus:

Replace identity by unifiability. Apply the mgu to the resulting clause. In the ordering restrictions, replace $\succ$ by $\not\preceq$.

However:

As in Knuth-Bendix completion, we do not want to consider overlaps at or below a variable position.

Consequence: there are inferences between ground instances $D\theta$ and $C\theta$ of clauses D and C which are *not* ground instances of inferences between D and C.

Such inferences have to be treated in a special way in the completeness proof.

The Superposition Calculus – Formally

Until now, we have seen most of the ideas behind the superposition calculus and its completeness proof.

We will now start again from the beginning giving precise definitions and proofs.

Inference rules:

Pos. Superposition:
$$\frac{D' \lor t \approx t' \qquad C' \lor s[u] \approx s'}{(D' \lor C' \lor s[t'] \approx s')\sigma}$$
where $\sigma = \mathrm{mgu}(t, u)$ and u is not a variable.

Neg. Superposition:
$$\frac{D' \lor t \approx t' \qquad C' \lor s[u] \not\approx s'}{(D' \lor C' \lor s[t'] \not\approx s')\sigma}$$
where $\sigma = \mathrm{mgu}(t, u)$ and u is not a variable.

Equality Resolution:
$$\frac{C' \lor s \not\approx s'}{C'\sigma}$$
where $\sigma = \mathrm{mgu}(s, s')$.

Equality Factoring:
$$\frac{C' \lor s' \approx t' \lor s \approx t}{(C' \lor t \not\approx t' \lor s \approx t')\sigma}$$
where $\sigma = \mathrm{mgu}(s, s')$.

Theorem 4.44 *All inference rules of the superposition calculus are* correct, *i. e., for every rule*
$$\frac{C_n, \ldots, C_1}{C_0}$$
we have $\{C_1, \ldots, C_n\} \models C_0$.

Proof. Exercise. □

Orderings:

Let $\succ$ be a *reduction ordering that is total on ground terms.*

To a positive literal $s \approx t$, we assign the multiset $\{s, t\}$, to a negative literal $s \not\approx t$ the multiset $\{s, s, t, t\}$. The *literal ordering* $\succ_L$ compares these multisets using the multiset extension of $\succ$.

The *clause ordering* $\succ_C$ compares clauses by comparing their multisets of literals using the multiset extension of $\succ_L$.

Inferences have to be computed only if the following ordering restrictions are satisfied:

– In superposition inferences, after applying the unifier to both premises, the left premise is not greater than or equal to the right one.

– The last literal in each premise is maximal in the respective premise, i. e., there exists no greater literal (strictly maximal for positive literals in superposition inferences, i. e., there exists no greater or equal literal).

– In these literals, the lhs is not smaller than the rhs (in superposition inferences: neither smaller nor equal).

A ground clause C is called *redundant w. r. t. a set of ground clauses* N, if it follows from clauses in N that are smaller than C.

A clause is *redundant w. r. t. a set of clauses* N, if all its ground instances are redundant w. r. t. $G_\Sigma(N)$.

The set of all clauses that are redundant w. r. t. N is denoted by $Red(N)$.

N is called *saturated up to redundancy*, if the conclusion of every inference from clauses in $N \setminus Red(N)$ is contained in $N \cup Red(N)$.

Superposition: Refutational Completeness

For a set E of ground equations, $\mathrm{T}_\Sigma(\emptyset)/E$ is an E-interpretation (or E-algebra) with universe $\{\, [t] \mid t \in \mathrm{T}_\Sigma(\emptyset) \,\}$.

One can show (similar to the proof of Birkhoff's Theorem) that for every *ground* equation $s \approx t$ we have $\mathrm{T}_\Sigma(\emptyset)/E \models s \approx t$ if and only if $s \leftrightarrow_E^* t$.

In particular, if E is a convergent set of rewrite rules R and $s \approx t$ is a ground equation, then $\mathrm{T}_\Sigma(\emptyset)/R \models s \approx t$ if and only if $s \downarrow_R t$. By abuse of terminology, we say that an equation or clause is valid (or true) in R if and only if it is true in $\mathrm{T}_\Sigma(\emptyset)/R$.