These complexity results motivate the study of subclasses of formulas (*fragments*) of first-order logic

Q: Can you think of any fragments of first-order logic for which validity is decidable?

Some Decidable Fragments

Some decidable fragments:

- *Monadic class*: no function symbols, all predicates unary; validity is NEXPTIME-complete.
- Variable-free formulas without equality: satisfiability is NP-complete. (why?)
- Variable-free Horn clauses (clauses with at most one positive atom): entailment is decidable in linear time.
- Finite model checking is decidable in time polynomial in the size of the structure and the formula.

3.5 Normal Forms and Skolemization (Traditional)

Study of normal forms motivated by

- reduction of logical concepts,
- efficient data structures for theorem proving.

The main problem in first-order logic is the treatment of quantifiers. The subsequent normal form transformations are intended to eliminate many of them.

Prenex Normal Form

Prenex formulas have the form

$$Q_1 x_1 \ldots Q_n x_n \; F,$$

where F is quantifier-free and $Q_i \in \{\forall, \exists\}$; we call $Q_1 x_1 \ldots Q_n x_n$ the *quantifier prefix* and F the *matrix* of the formula.

Computing prenex normal form by the rewrite relation $\Rightarrow_P$:

$$\begin{array}{rcll}
(F \leftrightarrow G) & \Rightarrow_P & (F \rightarrow G) \land (G \rightarrow F) & \\
\neg Q x F & \Rightarrow_P & \overline{Q} x \neg F & (\neg Q) \\
(Q x F \; \rho \; G) & \Rightarrow_P & Q y (F[y/x] \; \rho \; G), \; y \text{ fresh}, \; \rho \in \{\land, \lor\} & \\
(Q x F \rightarrow G) & \Rightarrow_P & \overline{Q} y (F[y/x] \rightarrow G), \; y \text{ fresh} & \\
(F \; \rho \; Q x G) & \Rightarrow_P & Q y (F \; \rho \; G[y/x]), \; y \text{ fresh}, \; \rho \in \{\land, \lor, \rightarrow\} &
\end{array}$$

Here $\overline{Q}$ denotes the quantifier *dual* to Q, i. e., $\overline{\forall} = \exists$ and $\overline{\exists} = \forall$.

Skolemization

Intuition: replacement of $\exists y$ by a concrete choice function computing y from all the arguments y depends on.

Transformation $\Rightarrow_S$ (to be applied outermost, *not* in subformulas):

$$\forall x_1, \ldots, x_n \exists y F \;\; \Rightarrow_S \;\; \forall x_1, \ldots, x_n F[f(x_1, \ldots, x_n)/y]$$

where f, where $\mathsf{arity}(f) = n$, is a new function symbol (*Skolem function*).

Together: $F \Rightarrow_P^* \underbrace{G}_{\text{prenex}} \Rightarrow_S^* \underbrace{H}_{\text{prenex, no } \exists}$

Theorem 3.9 *Let F, G, and H as defined above and closed. Then* *(i)* F *and* G *are equivalent.*

(ii) $H \models G$ *but the converse is not true in general.*

(iii) G *satisfiable (w. r. t.* Σ*-Alg)* $\Leftrightarrow$ H *satisfiable (w. r. t.* Σ'*-Alg) where* $\Sigma' = (\Omega \cup SKF, \Pi)$*, if* $\Sigma = (\Omega, \Pi)$*.*

Clausal Normal Form (Conjunctive Normal Form)

$$
\begin{array}{rcl}
(F \leftrightarrow G) & \Rightarrow_K & (F \rightarrow G) \wedge (G \rightarrow F) \\
(F \rightarrow G) & \Rightarrow_K & (\neg F \vee G) \\
\neg(F \vee G) & \Rightarrow_K & (\neg F \wedge \neg G) \\
\neg(F \wedge G) & \Rightarrow_K & (\neg F \vee \neg G) \\
\neg\neg F & \Rightarrow_K & F \\
(F \wedge G) \vee H & \Rightarrow_K & (F \vee H) \wedge (G \vee H) \\
(F \wedge \top) & \Rightarrow_K & F \\
(F \wedge \bot) & \Rightarrow_K & \bot \\
(F \vee \top) & \Rightarrow_K & \top \\
(F \vee \bot) & \Rightarrow_K & F
\end{array}
$$

These rules are to be applied modulo associativity and commutativity of $\wedge$ and $\vee$. The first five rules, plus the rule $(\neg Q)$, compute the *negation normal form* (NNF) of a formula.

The Complete Picture

$$
\begin{array}{llll}
F & \stackrel{*}{\Rightarrow}_P & Q_1 y_1 \ldots Q_n y_n \; G & (G \text{ quantifier-free}) \\
 & \stackrel{*}{\Rightarrow}_S & \forall x_1, \ldots, x_m \; H & (m \leq n,\ H \text{ quantifier-free}) \\
 & \stackrel{*}{\Rightarrow}_K & \underbrace{\underbrace{\forall x_1, \ldots, x_m}_{\text{leave out}} \; \bigwedge_{i=1}^{k} \underbrace{\bigvee_{j=1}^{n_i} L_{ij}}_{\text{clauses } C_i}}_{F'} &
\end{array}
$$

$N = \{C_1, \ldots, C_k\}$ is called the *clausal (normal) form* (CNF) of F.
Note: the variables in the clauses are implicitly universally quantified.

Theorem 3.10 *Let* F *be closed. Then* $F' \models F$*. (The converse is not true in general.)*

Theorem 3.11 *Let* F *be closed. Then* F *is satisfiable iff* F' *is satisfiable iff* N *is satisfiable*