Exercise Problems for “Automated Reasoning”

Exercise 1 (Propositional Logic)
For a propositional formula $X$, let $b(X)$ be the number of occurrences of connectives ($\land$, $\lor$) in $X$. Prove using structural induction that, for every formula $X$, $b(X) =$ number of left parentheses in $X =$ number of right parentheses in $X$. (Hence every propositional formula has the same number of left as right parentheses.)

Exercise 2 (DPLL)
Prove Lemma 1.9 by induction on the length of the derivation.

Exercise 3 (DPLL)
Let $N$ be the following set of propositional clauses:

\[
\begin{align*}
- P & \lor - Q \lor R \lor - S \\
- Q & \lor - R \\
Q & \lor - R \lor S \\
- P & \lor R \\
- P & \lor Q \lor - S \\
 & \lor S
\end{align*}
\]

During a DPLL-derivation, we have reached the configuration $P^dQ^d\rightarrow \parallel N$.

a) Give two different backjump clauses that can be used in this situation.

b) Select one backjump clause and continue until you find a model or until you reach fail.

Exercise 4 (Miniscoping)
Apply miniscoping to the following formula:

\[
\forall x \exists y \forall z (R(x, x) \land [P(y) \lor R(x, y) \lor Q(z)])
\]
Exercise 5 \((DPLL(LA))\)
Use the transition system \(\Rightarrow_{DPLL(T)}\) for the DPLL(T) procedure to check whether the fol-
lowing formula \(F\) is satisfiable or not. For each step explain briefly which rule you use and why.
Check the \(T\)-(in)consistency of a valuation by using the Fourier-Motzkin algorithm.

\[ F = \neg((x \geq 5 \lor x < y) \land 3x + y \geq 20 \land y = x + 3) \rightarrow y \geq 9 \]

Exercise 6 \((Unification)\)
Solve the following unification problems using both \(\Rightarrow_{SU}\) and \(\Rightarrow_{PU}\).

1. \(E_1 = \{ f(g(y, x_3), y, g(k(x_3), a)) \doteq f(x_1, h(x_2), x_1) \}\)
2. \(E_2 = \{ f(x_1, h(y, g(y)), h(x_1, a)) \doteq f(g(x_2), z, z) \}\)
3. \(E_3 = \{ f(a, x_1, g(h(y, a))) \doteq f(z, g(z), g(x_2)) \}\)

Exercise 7 \((Knuth-Bendix Completion)\)
Apply the Knuth-Bendix completion procedure to the set of equations

\[ E_0 = \{ f(x, y) \approx h(x, c), f(x, y) \approx h(c, y), h(c, c) \approx c \} \]

using the lexicographic path order with the precedence \(f > h > c\).

Exercise 8 \((Knuth-Bendix Completion)\)
Show that the strict part \(\sqsubset = \sqsubset \setminus \sim\) of the encompassment quasi-ordering \(\sqsubset\) is a well-founded
strict partial ordering.

Exercise 9 \((Orderings)\)
For each of the following term rewriting systems (1) to (4) give, if possible, an instance of a
lexicographic path ordering and a Knuth-Bendix ordering such that the left-hand sides are
larger than the right-hand sides when you compare the rules. If the rule cannot be ordered
using KBO/LPO give a short argument why it can’t.

(1) \(f(g(x), g(y)) \rightarrow f(f(x, g(y)), x)\);
(2) \(f(g(x, a), x, a) \rightarrow g(f(x, x, a), h(b, x))\);
(3) \(f(x, g(y), a) \rightarrow f(h(x), h(y), h(a))\);
(4) \(x + 0 \rightarrow x\)
\(x + s(y) \rightarrow s(x + y)\).
**Exercise 10 (Superposition)**

Use the superposition calculus in order to do all possible inferences between the given clauses. Use KBO with weight 1 for all function symbols and variables and the precedence \( g > f > c > b > a \).

1. \( f(f(x, y), y) \nless x \) and \( f(a, b) \approx a \lor a \nless b \)
2. \( f(x, y) \nless g(y) \lor g(x) \nless f(y, x) \) and \( f(a, b) \approx a \lor g(a) \approx b \)
3. \( f(a, y) \approx f(x, b) \lor f(x, b) \nless b \) and \( f(x, y) \approx x \)
4. \( a \approx b \) and \( g(b) \approx a \lor g(b) \approx g(a) \)

**Exercise 11 (Superposition)**

Consider the following clause set \( N \):

1. \( f(a, b) \approx b \)
2. \( f(y, x) \nless y \)
3. \( f(f(x, y), f(y, x)) \nless x \lor g(x, y) \approx x \)

1. Apply Superposition to saturate the clause set. Use KBO with weight 1 for all function symbols and variables and the precedence \( g > f > c > b > a \).
2. Check yourself using SPASS: write the corresponding input file and call SPASS with the options “-PDer”, “-DocProof”, “-Select=0”. Compare your result with the result of SPASS.

**Exercise 12 (Candidate Interpretation, Unit Rewriting)**

Consider the following clause set \( N \). Use KBO with weight 1 for all function symbols and variables and the precedence \( f > g > b > a \).

1. \( a \approx b \)
2. \( f(a, b) \approx a \)
3. \( f(b, a) \nless b \lor g(a) \approx b \)

1. Which clause is false in \( R_\infty \)?
2. Saturate the clauses using unit rewriting.
3. Repeat the construction of the candidate interpretation.
Exercise 13 (Ordered Resolution with Selection)
Refute the following clauses using the calculus of ordered resolution with selection. Take a lexicographic path order and the precedence \( f > h > g > Q > R > P > a. \)

1. \( \{R(x_1, f(a)), \neg P(g(x_1, a), y), R(h(y), f(z))\} \)
2. \( \{\neg R(x_2, x_3), Q(g(h(x_2), x_3))\} \)
3. \( \{P(g(h(x_4), x_4), x_4), Q(g(h(h(x_4))), z)\} \)
4. \( \{\neg Q(g(h(z)), f(z))\} \)