2.5 Normal Forms and Skolemization (Traditional)

Study of normal forms motivated by

- reduction of logical concepts,
- efficient data structures for theorem proving.

The main problem in first-order logic is the treatment of quantifiers. The subsequent normal form transformations are intended to eliminate many of them.

Prenex Normal Form

Prenex formulas have the form

$$Q_1 x_1 \dots Q_n x_n \; F,$$

where F is quantifier-free and $Q_i \in \{\forall, \exists\}$; we call $Q_1 x_1 \dots Q_n x_n$ the *quantifier prefix* and F the *matrix* of the formula.

Computing prenex normal form by the rewrite relation $\Rightarrow_P$:

$$\begin{array}{rcll} (F \leftrightarrow G) & \Rightarrow_P & (F \rightarrow G) \wedge (G \rightarrow F) & \\ \neg QxF & \Rightarrow_P & \overline{Q}x\neg F & (\neg Q) \\ (QxF \; \rho \; G) & \Rightarrow_P & Qy(F[y/x] \; \rho \; G), \; y \text{ fresh}, \; \rho \in \{\wedge, \vee\} & \\ (QxF \rightarrow G) & \Rightarrow_P & \overline{Q}y(F[y/x] \rightarrow G), \; y \text{ fresh} & \\ (F \; \rho \; QxG) & \Rightarrow_P & Qy(F \; \rho \; G[y/x]), \; y \text{ fresh}, \; \rho \in \{\wedge, \vee, \rightarrow\} & \end{array}$$

Here $\overline{Q}$ denotes the quantifier *dual* to Q, i. e., $\overline{\forall} = \exists$ and $\overline{\exists} = \forall$.

Skolemization

Intuition: replacement of $\exists y$ by a concrete choice function computing y from all the arguments y depends on.

Transformation $\Rightarrow_S$ (to be applied outermost, *not* in subformulas):

$$\forall x_1, \dots, x_n \exists y F \;\; \Rightarrow_S \;\; \forall x_1, \dots, x_n F[f(x_1, \dots, x_n)/y]$$

where f, where $\mathsf{arity}(f) = n$, is a new function symbol (*Skolem function*).

Together: $F \overset{*}{\Rightarrow}_P \underbrace{G}_{\text{prenex}} \overset{*}{\Rightarrow}_S \underbrace{H}_{\text{prenex, no } \exists}$

Theorem 2.9 *Let F, G, and H as defined above and closed. Then* *(i)* F *and* G *are equivalent.*

(ii) $H \models G$ *but the converse is not true in general.*

(iii) G *satisfiable (w. r. t.* Σ*-Alg)* $\Leftrightarrow$ H *satisfiable (w. r. t.* Σ'*-Alg) where* $\Sigma' = (\Omega \cup SKF, \Pi)$*, if* $\Sigma = (\Omega, \Pi)$*.*

Clausal Normal Form (Conjunctive Normal Form)

$$
\begin{array}{rcl}
(F \leftrightarrow G) & \Rightarrow_K & (F \rightarrow G) \wedge (G \rightarrow F) \\
(F \rightarrow G) & \Rightarrow_K & (\neg F \vee G) \\
\neg(F \vee G) & \Rightarrow_K & (\neg F \wedge \neg G) \\
\neg(F \wedge G) & \Rightarrow_K & (\neg F \vee \neg G) \\
\neg\neg F & \Rightarrow_K & F \\
(F \wedge G) \vee H & \Rightarrow_K & (F \vee H) \wedge (G \vee H) \\
(F \wedge \top) & \Rightarrow_K & F \\
(F \wedge \bot) & \Rightarrow_K & \bot \\
(F \vee \top) & \Rightarrow_K & \top \\
(F \vee \bot) & \Rightarrow_K & F
\end{array}
$$

These rules are to be applied modulo associativity and commutativity of $\wedge$ and $\vee$. The first five rules, plus the rule $(\neg Q)$, compute the *negation normal form* (NNF) of a formula.

The Complete Picture

$$
\begin{array}{rll}
F & \stackrel{*}{\Rightarrow}_P \; Q_1 y_1 \ldots Q_n y_n \; G & (G \text{ quantifier-free}) \\
& \stackrel{*}{\Rightarrow}_S \; \forall x_1, \ldots, x_m \; H & (m \leq n,\ H \text{ quantifier-free}) \\
& \stackrel{*}{\Rightarrow}_K \; \underbrace{\underbrace{\forall x_1, \ldots, x_m}_{\text{leave out}} \bigwedge_{i=1}^{k} \underbrace{\bigvee_{j=1}^{n_i} L_{ij}}_{\text{clauses } C_i}}_{F'} &
\end{array}
$$

$N = \{C_1, \ldots, C_k\}$ is called the *clausal (normal) form* (CNF) of F.
Note: the variables in the clauses are implicitly universally quantified.

Theorem 2.10 *Let* F *be closed. Then* $F' \models F$*. (The converse is not true in general.)*

Theorem 2.11 *Let* F *be closed. Then* F *is satisfiable iff* F' *is satisfiable iff* N *is satisfiable*

Optimization

Here is lots of room for optimization since we only can preserve satisfiability anyway:

- size of the CNF exponential when done naively;
  but see the transformations we introduced for propositional logic
- want to preserve the original formula structure;
- want small arity of Skolem functions (follows)

2.6 Getting small Skolem Functions

- produce a negation normal form (NNF)
- apply miniscoping
- rename all variables
- skolemize

Negation Normal Form (NNF)

Apply the rewrite relation $\Rightarrow_{NNF}$, F is the overall formula:

$$
\begin{array}{rcl}
G \leftrightarrow H & \Rightarrow_{NNF} & (G \rightarrow H) \land (H \rightarrow G) \\
\multicolumn{3}{l}{\qquad \text{if } F/p = G \leftrightarrow H \text{ and } F/p \text{ has positive polarity}} \\
G \leftrightarrow H & \Rightarrow_{NNF} & (G \land H) \lor (\neg H \land \neg G) \\
\multicolumn{3}{l}{\qquad \text{if } F/p = G \leftrightarrow H \text{ and } F/p \text{ has negative polarity}} \\
\neg Qx\, G & \Rightarrow_{NNF} & \overline{Q}x \neg G \\
\neg(G \lor H) & \Rightarrow_{NNF} & \neg G \land \neg H \\
\neg(G \land H) & \Rightarrow_{NNF} & \neg G \lor \neg H \\
G \rightarrow H & \Rightarrow_{NNF} & \neg G \lor H \\
\neg\neg G & \Rightarrow_{NNF} & G
\end{array}
$$

Miniscoping

Apply the rewrite relation $\Rightarrow_{MS}$. For the below rules we assume that x occurs freely in G, H, but x does not occur freely in F:

$$
\begin{array}{rcl}
Qx\,(G \land F) & \Rightarrow_{MS} & Qx\, G \land F \\
Qx\,(G \lor F) & \Rightarrow_{MS} & Qx\, G \lor F \\
\forall x\,(G \land H) & \Rightarrow_{MS} & \forall x\, G \land \forall x\, H \\
\exists x\,(G \lor H) & \Rightarrow_{MS} & \exists x\, G \lor \exists x\, H
\end{array}
$$

Variable Renaming

Rename all variables in F such that there are no two different positions p, q with $F/p = Qx\, G$ and $F/q = Qx\, H$.

Standard Skolemization

Let F be the overall formula, then apply the rewrite rule:

$$\begin{array}{ll} \exists x\, H & \Rightarrow_{SK} \quad H[f(y_1, \ldots, y_n)/x] \\ & \text{if } F/p = \exists x\, H \text{ and } p \text{ has minimal length,} \\ & \{y_1, \ldots, y_n\} \text{ are the free variables in } \exists x\, H, \\ & f \text{ is a new function symbol, } \mathsf{arity}(f) = n \end{array}$$

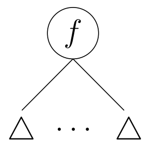
2.7 Herbrand Interpretations

From now an we shall consider PL without equality. Ω shall contains at least one constant symbol.

A *Herbrand interpretation* (over Σ) is a Σ-algebra $\mathcal{A}$ such that

- $U_{\mathcal{A}} = \mathrm{T}_{\Sigma}$ (= the set of ground terms over Σ)
- $f_{\mathcal{A}} : (s_1, \ldots, s_n) \mapsto f(s_1, \ldots, s_n),\ f \in \Omega, \mathsf{arity}(f) = n$

$f_{\mathcal{A}}(\triangle, \ldots, \triangle) =$



In other words, *values are fixed* to be ground terms and *functions are fixed* to be the *term constructors*. Only predicate symbols $p \in \Pi$, $\mathsf{arity}(p) = m$ may be freely interpreted as relations $p_{\mathcal{A}} \subseteq \mathrm{T}_{\Sigma}^{m}$.

Proposition 2.12 *Every set of ground atoms I uniquely determines a Herbrand interpretation $\mathcal{A}$ via*

$$(s_1, \ldots, s_n) \in p_{\mathcal{A}} \quad :\Leftrightarrow \quad p(s_1, \ldots, s_n) \in I$$

Thus we shall identify Herbrand interpretations (over Σ) with sets of Σ-ground atoms.

Example: $\Sigma_{Pres} = (\{0/0, s/1, +/2\},\ \{</2, \leq/2\})$

$\mathbb{N}$ as Herbrand interpretation over Σ_{Pres}:

$$
\begin{array}{ll}
I = \{ & 0 \leq 0,\ 0 \leq s(0),\ 0 \leq s(s(0)),\ \ldots, \\
 & 0+0 \leq 0,\ 0+0 \leq s(0),\ \ldots, \\
 & \ldots,\ (s(0)+0)+s(0) \leq s(0)+(s(0)+s(0)) \\
 & \ldots \\
 & s(0)+0 < s(0)+0+0+s(0) \\
 & \ldots\}
\end{array}
$$

Existence of Herbrand Models

A Herbrand interpretation I is called a *Herbrand model* of F, if $I \models F$.

Theorem 2.13 (Herbrand) *Let N be a set of Σ-clauses.*

$$
\begin{array}{lll}
N \text{ satisfiable} & \Leftrightarrow & N \text{ has a Herbrand model (over } \Sigma) \\
 & \Leftrightarrow & G_\Sigma(N) \text{ has a Herbrand model (over } \Sigma)
\end{array}
$$

where $G_\Sigma(N) = \{C\sigma \text{ ground clause} \mid C \in N,\ \sigma : X \to \mathrm{T}_\Sigma\}$ *is the set of* ground instances *of N.*

[The proof will be given below in the context of the completeness proof for resolution.]

Example of a G_Σ

For Σ_{Pres} one obtains for

$$C = (x < y) \vee (y \leq s(x))$$

the following ground instances:

$(0 < 0) \vee (0 \leq s(0))$
$(s(0) < 0) \vee (0 \leq s(s(0)))$
$\ldots$
$(s(0)+s(0) < s(0)+0) \vee (s(0)+0 \leq s(s(0)+s(0)))$
$\ldots$

2.8 Inference Systems and Proofs

Inference systems Γ (proof calculi) are sets of tuples

$$(F_1, \dots, F_n, F_{n+1}),\ n \geq 0,$$

called *inferences* or *inference rules*, and written

$$\frac{\overbrace{F_1 \ \dots \ F_n}^{\text{premises}}}{\underbrace{F_{n+1}}_{\text{conclusion}}}\ .$$

Clausal inference system: premises and conclusions are clauses. One also considers inference systems over other data structures (cf. below).

Proofs

A *proof* in Γ of a formula F from a a set of formulas N (called *assumptions*) is a sequence $F_1, \dots, F_k$ of formulas where

(i) $F_k = F$,

(ii) for all $1 \leq i \leq k$: $F_i \in N$, or else there exists an inference

$$\frac{F_{i_1} \ \dots \ F_{i_{n_i}}}{F_i}$$

in Γ, such that $0 \leq i_j < i$, for $1 \leq j \leq n_i$.

Soundness and Completeness

Provability $\vdash_\Gamma$ of F from N in Γ: $N \vdash_\Gamma F \ \ :\Leftrightarrow$ there exists a proof Γ of F from N.

Γ is called *sound* $\ :\Leftrightarrow$

$$\frac{F_1 \ \dots \ F_n}{F} \in \Gamma \quad \Rightarrow \quad F_1, \dots, F_n \models F$$

Γ is called *complete* $\ :\Leftrightarrow$

$$N \models F \quad \Rightarrow \quad N \vdash_\Gamma F$$

Γ is called *refutationally complete* $\ :\Leftrightarrow$

$$N \models \bot \quad \Rightarrow \quad N \vdash_\Gamma \bot$$

Proposition 2.14

(i) *Let Γ be sound. Then $N \vdash_\Gamma F \Rightarrow N \models F$*

(ii) *$N \vdash_\Gamma F \Rightarrow$ there exist $F_1, \ldots, F_n \in N$ s.t. $F_1, \ldots, F_n \vdash_\Gamma F$ (resembles compactness).*

Proofs as Trees

markings	$\widehat{=}$	formulas			
leaves	$\widehat{=}$	assumptions and axioms			
other nodes	$\widehat{=}$	inferences:	conclusion	$\widehat{=}$	ancestor
			premises	$\widehat{=}$	direct descendants

$$\dfrac{P(f(c)) \qquad \dfrac{\dfrac{\dfrac{P(f(c)) \lor Q(b) \qquad \dfrac{\dfrac{P(f(c)) \lor Q(b) \qquad \neg P(f(c)) \lor \neg P(f(c)) \lor Q(b)}{\neg P(f(c)) \lor Q(b) \lor Q(b)}}{\neg P(f(c)) \lor Q(b)}}{Q(b) \lor Q(b)}}{Q(b)} \qquad \neg P(f(c)) \lor \neg Q(b)}{\neg P(f(c))}}{\bot}$$

2.9 Propositional Resolution

We observe that propositional clauses and ground clauses are the same concept.

In this section we only deal with ground clauses.

The Resolution Calculus *Res*

Resolution inference rule:

$$\frac{D \lor A \qquad \neg A \lor C}{D \lor C}$$

Terminology: $D \lor C$: *resolvent*; A: *resolved atom*

(Positive) factorisation inference rule:

$$\frac{C \lor A \lor A}{C \lor A}$$

These are *schematic inference rules*; for each substitution of the *schematic variables* C, D, and A, respectively, by ground clauses and ground atoms we obtain an inference rule.

As "$\lor$" is considered associative and commutative, we assume that A and $\neg A$ can occur anywhere in their respective clauses.

Sample Refutation

1.	$\neg P(f(c)) \vee \neg P(f(c)) \vee Q(b)$	(given)
2.	$P(f(c)) \vee Q(b)$	(given)
3.	$\neg P(g(b,c)) \vee \neg Q(b)$	(given)
4.	$P(g(b,c))$	(given)
5.	$\neg P(f(c)) \vee Q(b) \vee Q(b)$	(Res. 2. into 1.)
6.	$\neg P(f(c)) \vee Q(b)$	(Fact. 5.)
7.	$Q(b) \vee Q(b)$	(Res. 2. into 6.)
8.	$Q(b)$	(Fact. 7.)
9.	$\neg P(g(b,c))$	(Res. 8. into 3.)
10.	$\bot$	(Res. 4. into 9.)

Resolution with Implicit Factorization RIF

$$\frac{D \vee A \vee \ldots \vee A \qquad \neg A \vee C}{D \vee C}$$

1.	$\neg P(f(c)) \vee \neg P(f(c)) \vee Q(b)$	(given)
2.	$P(f(c)) \vee Q(b)$	(given)
3.	$\neg P(g(b,c)) \vee \neg Q(b)$	(given)
4.	$P(g(b,c))$	(given)
5.	$\neg P(f(c)) \vee Q(b) \vee Q(b)$	(Res. 2. into 1.)
6.	$Q(b) \vee Q(b) \vee Q(b)$	(Res. 2. into 5.)
7.	$\neg P(g(b,c))$	(Res. 6. into 3.)
8.	$\bot$	(Res. 4. into 7.)

Soundness of Resolution

Theorem 2.15 *Propositional resolution is sound.*

Proof. Let $I \in \Sigma$-Alg. To be shown:

(i) for resolution: $I \models D \vee A,\ I \models C \vee \neg A \ \Rightarrow\ I \models D \vee C$

(ii) for factorization: $I \models C \vee A \vee A \ \Rightarrow\ I \models C \vee A$

(i): Assume premises are valid in I. Two cases need to be considered:
If $I \models A$, then $I \models C$, hence $I \models D \vee C$.
Otherwise, $I \models \neg A$, then $I \models D$, and again $I \models D \vee C$.
(ii): even simpler. □

Note: In propositional logic (ground clauses) we have:

1. $I \models L_1 \vee \ldots \vee L_n \;\Leftrightarrow\;$ there exists i: $I \models L_i$.
2. $I \models A$ or $I \models \neg A$.

This does not hold for formulas with variables!