2.10 Well-Founded Orderings

Literature: Franz Baader and Tobias Nipkow: *Term rewriting and all that*, Cambridge Univ. Press, 1998, Chapter 2.

To show the refutational completeness of resolution, we will make use of the concept of well-founded orderings.

Partial Orderings

A *(strict) partial ordering* $\succ$ on a set M is a transitive and irreflexive binary relation on M.

An $a \in M$ is called *minimal*, if there is no b in M such that $a \succ b$.

An $a \in M$ is called *smallest*, if $b \succ a$ for all $b \in M$ different from a.

Notation:
$\prec$ for the inverse relation $\succ^{-1}$
$\succeq$ for the reflexive closure $(\succ \cup =)$ of $\succ$

Well-Foundedness

A (strict) partial ordering $\succ$ is called *well-founded (Noetherian)*, if there is no infinite descending chain $a_0 \succ a_1 \succ a_2 \succ \ldots$ with $a_i \in M$.

Well-Founded Orderings: Examples

Natural numbers. $(\mathbb{N}, >)$

Lexicographic orderings. Let $(M_1, \succ_1), (M_2, \succ_2)$ be well-founded orderings. Then let their *lexicographic combination*

$$\succ = (\succ_1, \succ_2)_{lex}$$

on $M_1 \times M_2$ be defined as

$$(a_1, a_2) \succ (b_1, b_2) \quad :\Leftrightarrow \quad a_1 \succ_1 b_1, \text{ or else } a_1 = b_1 \;\&\; a_2 \succ_2 b_2$$

(analogously for more than two orderings)

This again yields a well-founded ordering (proof below).

Length-based ordering on words. For alphabets Σ with a well-founded ordering $>_\Sigma$, the relation $\succ$, defined as

$$w \succ w' \quad := \quad \alpha)\ |w| > |w'| \text{ or}$$
$$\beta)\ |w| = |w'| \text{ and } w >_{\Sigma,lex} w',$$

is a well-founded ordering on Σ^* (proof below).

Counterexamples:

$(\mathbb{Z}, >)$;

$(\mathbb{N}, <)$;

the lexicographic ordering on Σ^*

Basic Properties of Well-Founded Orderings

Lemma 2.16 *$(M, \succ)$ is well-founded if and only if every $\emptyset \subset M' \subseteq M$ has a minimal element.*

Lemma 2.17 *$(M_i, \succ_i)$ is well-founded for $i = 1, 2$ if and only if $(M_1 \times M_2,\ \succ)$ with $\succ = (\succ_1, \succ_2)_{lex}$ is well-founded.*

Proof. (i) "$\Rightarrow$": Suppose $(M_1 \times M_2,\ \succ)$ is not well-founded. Then there is an infinite sequence $(a_0, b_0) \succ (a_1, b_1) \succ (a_2, b_2) \succ \ldots$.

Let $A = \{a_i \mid i \geq 0\} \subseteq M_1$. Since $(M_1, \succ_1)$ is well-founded, A has a minimal element a_n. But then $B = \{b_i \mid i \geq n\} \subseteq M_2$ can not have a minimal element, contradicting the well-foundedness of $(M_2, \succ_2)$.

(ii) "$\Leftarrow$": obvious. □

Noetherian Induction

Theorem 2.18 (Noetherian Induction) *Let $(M, \succ)$ be a well-founded ordering, let Q be a property of elements of M.*

If for all $m \in M$ the implication

if $Q(m')$, for all $m' \in M$ such that $m \succ m'$,[1]
then $Q(m)$.[2]

is satisfied, then the property $Q(m)$ holds for all $m \in M$.

[1] induction hypothesis
[2] induction step

Proof. Let $X = \{m \in M \mid Q(m) \text{ false}\}$. Suppose, $X \neq \emptyset$. Since $(M, \succ)$ is well-founded, X has a minimal element m_1. Hence for all $m' \in M$ with $m' \prec m_1$ the property $Q(m')$ holds. On the other hand, the implication which is presupposed for this theorem holds in particular also for m_1, hence $Q(m_1)$ must be true so that m_1 can not be in X. *Contradiction.* □

Multi-Sets

Let M be a set. A *multi-set* S over M is a mapping $S : M \to \mathbb{N}$. Hereby $S(m)$ specifies the number of occurrences of elements m of the base set M within the multi-set S.

We say that m is an *element* of S, if $S(m) > 0$.

We use set notation ($\in$, $\subset$, $\subseteq$, $\cup$, $\cap$, etc.) with analogous meaning also for multi-sets, e. g.,

$$
\begin{aligned}
(S_1 \cup S_2)(m) &= S_1(m) + S_2(m) \\
(S_1 \cap S_2)(m) &= \min\{S_1(m), S_2(m)\}
\end{aligned}
$$

A multi-set is called *finite*, if

$$|\{m \in M \mid s(m) > 0\}| < \infty,$$

for each m in M.

From now on we only consider finite multi-sets.

Example. $S = \{a, a, a, b, b\}$ is a multi-set over $\{a, b, c\}$, where $S(a) = 3$, $S(b) = 2$, $S(c) = 0$.

Multi-Set Orderings

Lemma 2.19 (König's Lemma) *Every finitely branching tree with infinitely many nodes contains an infinite path.*

Let $(M, \succ)$ be a partial ordering. The *multi-set extension* of $\succ$ to multi-sets over M is defined by

$$
\begin{aligned}
S_1 \succ_{\text{mul}} S_2 :\Leftrightarrow\ & S_1 \neq S_2 \\
& \text{and } \forall m \in M : [S_2(m) > S_1(m) \\
& \qquad \Rightarrow \quad \exists m' \in M : (m' \succ m \text{ and } S_1(m') > S_2(m'))]
\end{aligned}
$$

Theorem 2.20
(a) $\succ_{\text{mul}}$ *is a partial ordering.*
(b) $\succ$ *well-founded* $\Rightarrow \succ_{\text{mul}}$ *well-founded.*
(c) $\succ$ *total* $\Rightarrow \succ_{\text{mul}}$ *total.*

Proof. see Baader and Nipkow, page 22–24. □

2.11 Refutational Completeness of Resolution

How to show refutational completeness of propositional resolution:

- We have to show: $N \models \bot \;\Rightarrow\; N \vdash_{Res} \bot$, or equivalently: If $N \not\vdash_{Res} \bot$, then N has a model.
- Idea: Suppose that we have computed sufficiently many inferences (and not derived $\bot$).
- Now order the clauses in N according to some appropriate ordering, inspect the clauses in ascending order, and construct a series of Herbrand interpretations.
- The limit interpretation can be shown to be a model of N.

Clause Orderings

1. We assume that $\succ$ is any fixed ordering on ground atoms that is *total* and *well-founded*. (There exist many such orderings, e. g., the lenght-based ordering on atoms when these are viewed as words over a suitable alphabet.)
2. Extend $\succ$ to an *ordering* $\succ_L$ *on ground literals:*

$$\begin{array}{lcl} [\neg]A & \succ_L & [\neg]B \quad \text{, if } A \succ B \\ \neg A & \succ_L & A \end{array}$$

3. Extend $\succ_L$ to an *ordering* $\succ_C$ *on ground clauses:*
   $\succ_C = (\succ_L)_{\text{mul}}$, the multi-set extension of $\succ_L$.

   Notation: $\succ$ also for $\succ_L$ and $\succ_C$.

Example

Suppose $A_5 \succ A_4 \succ A_3 \succ A_2 \succ A_1 \succ A_0$. Then:

$$\begin{array}{cc} & A_0 \lor A_1 \\ \prec & A_1 \lor A_2 \\ \prec & \neg A_1 \lor A_2 \\ \prec & \neg A_1 \lor A_4 \lor A_3 \\ \prec & \neg A_1 \lor \neg A_4 \lor A_3 \\ \prec & \neg A_5 \lor A_5 \end{array}$$

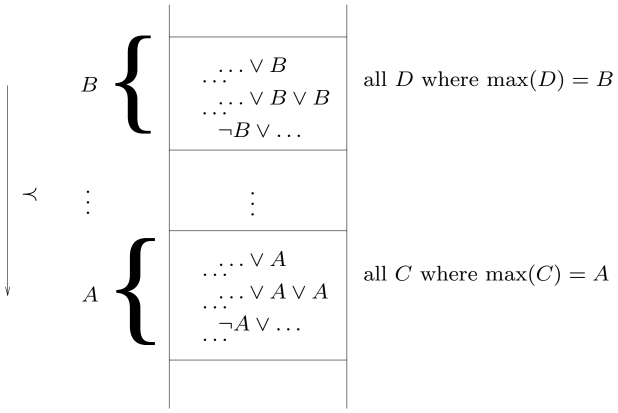
Properties of the Clause Ordering

Proposition 2.21

1. *The orderings on literals and clauses are total and well-founded.*
2. *Let C and D be clauses with $A = \max(C)$, $B = \max(D)$, where $\max(C)$ denotes the maximal atom in C.*
   (i) If $A \succ B$ then $C \succ D$.
   (ii) If $A = B$, A occurs negatively in C but only positively in D, then $C \succ D$.

Stratified Structure of Clause Sets

Let $A \succ B$. Clause sets are then stratified in this form:



Closure of Clause Sets under Res

$$
\begin{aligned}
Res(N) &= \{C \mid C \text{ is concl. of a rule in } Res \text{ w/ premises in } N\} \\
Res^0(N) &= N \\
Res^{n+1}(N) &= Res(Res^n(N)) \cup Res^n(N), \text{ for } n \geq 0 \\
Res^*(N) &= \textstyle\bigcup_{n\geq 0} Res^n(N)
\end{aligned}
$$

N is called *saturated* (w. r. t. resolution), if $Res(N) \subseteq N$.

Proposition 2.22

(i) $Res^*(N)$ *is saturated.*

(ii) Res *is refutationally complete, iff for each set* N *of ground clauses:*

$$N \models \bot \;\Leftrightarrow\; \bot \in Res^*(N)$$

Construction of Interpretations

Given: set N of ground clauses, atom ordering $\succ$.
Wanted: Herbrand interpretation I such that

- "many" clauses from N are valid in I;
- $I \models N$, *if* N *is saturated and* $\bot \notin N$.

Construction according to $\succ$, starting with the minimal clause.

Example

Let $A_5 \succ A_4 \succ A_3 \succ A_2 \succ A_1 \succ A_0$ (max. literals in red)

	clauses C	I_C	Δ_C	Remarks
1	$\neg A_0$	$\emptyset$	$\emptyset$	true in I_C
2	$A_0 \vee A_1$	$\emptyset$	$\{A_1\}$	A_1 maximal
3	$A_1 \vee A_2$	$\{A_1\}$	$\emptyset$	true in I_C
4	$\neg A_1 \vee A_2$	$\{A_1\}$	$\{A_2\}$	A_2 maximal
5	$\neg A_1 \vee A_4 \vee A_3 \vee A_0$	$\{A_1, A_2\}$	$\{A_4\}$	A_4 maximal
6	$\neg A_1 \vee \neg A_4 \vee A_3$	$\{A_1, A_2, A_4\}$	$\emptyset$	A_3 not maximal; *min. counter-ex.*
7	$\neg A_1 \vee A_5$	$\{A_1, A_2, A_4\}$	$\{A_5\}$	

$I = \{A_1, A_2, A_4, A_5\}$ is not a model of the clause set
$\Rightarrow$ there exists a *counterexample*.

Main Ideas of the Construction

- Clauses are considered in the order given by $\prec$.
- When considering C, one already has a partial interpretation I_C (initially $I_C = \emptyset$) available.
- If C is true in the partial interpretation I_C, nothing is done. ($\Delta_C = \emptyset$).
- If C is false, one would like to change I_C such that C becomes true.
- Changes should, however, be *monotone*. One never deletes anything from I_C and the truth value of clauses smaller than C should be maintained the way it was in I_C.
- Hence, one chooses $\Delta_C = \{A\}$ if, and only if, C is false in I_C, if A occurs positively in C (*adding A will make C become true*) and if this occurrence in C is strictly maximal in the ordering on literals (*changing the truth value of A has no effect on smaller clauses*).

Resolution Reduces Counterexamples

$$\frac{\neg A_1 \lor A_4 \lor A_3 \lor A_0 \qquad \neg A_1 \lor \neg A_4 \lor A_3}{\neg A_1 \lor \neg A_1 \lor A_3 \lor A_3 \lor A_0}$$

Construction of I for the extended clause set:

clauses C	I_C	Δ_C	Remarks
$\neg A_0$	$\emptyset$	$\emptyset$	
$A_0 \lor A_1$	$\emptyset$	$\{A_1\}$	
$A_1 \lor A_2$	$\{A_1\}$	$\emptyset$	
$\neg A_1 \lor A_2$	$\{A_1\}$	$\{A_2\}$	
$\neg A_1 \lor \neg A_1 \lor A_3 \lor A_3 \lor A_0$	$\{A_1, A_2\}$	$\emptyset$	A_3 occurs twice *minimal counter-ex.*
$\neg A_1 \lor A_4 \lor A_3 \lor A_0$	$\{A_1, A_2\}$	$\{A_4\}$	
$\neg A_1 \lor \neg A_4 \lor A_3$	$\{A_1, A_2, A_4\}$	$\emptyset$	counterexample
$\neg A_1 \lor A_5$	$\{A_1, A_2, A_4\}$	$\{A_5\}$	

The same I, but smaller counterexample, hence some progress was made.

Factorization Reduces Counterexamples

$$\frac{\neg A_1 \vee \neg A_1 \vee A_3 \vee A_3 \vee A_0}{\neg A_1 \vee \neg A_1 \vee A_3 \vee A_0}$$

Construction of I for the extended clause set:

clauses C	I_C	Δ_C	Remarks
$\neg A_0$	$\emptyset$	$\emptyset$	
$A_0 \vee A_1$	$\emptyset$	$\{A_1\}$	
$A_1 \vee A_2$	$\{A_1\}$	$\emptyset$	
$\neg A_1 \vee A_2$	$\{A_1\}$	$\{A_2\}$	
$\neg A_1 \vee \neg A_1 \vee A_3 \vee A_0$	$\{A_1, A_2\}$	$\{A_3\}$	
$\neg A_1 \vee \neg A_1 \vee A_3 \vee A_3 \vee A_0$	$\{A_1, A_2, A_3\}$	$\emptyset$	true in I_C
$\neg A_1 \vee A_4 \vee A_3 \vee A_0$	$\{A_1, A_2, A_3\}$	$\emptyset$	
$\neg A_1 \vee \neg A_4 \vee A_3$	$\{A_1, A_2, A_3\}$	$\emptyset$	true in I_C
$\neg A_3 \vee A_5$	$\{A_1, A_2, A_3\}$	$\{A_5\}$	

The resulting $I = \{A_1, A_2, A_3, A_5\}$ is a model of the clause set.

Construction of Candidate Interpretations

Let $N, \succ$ be given. We define sets I_C and Δ_C for all ground clauses C over the given signature inductively over $\succ$:

$$I_C := \bigcup_{C \succ D} \Delta_D$$

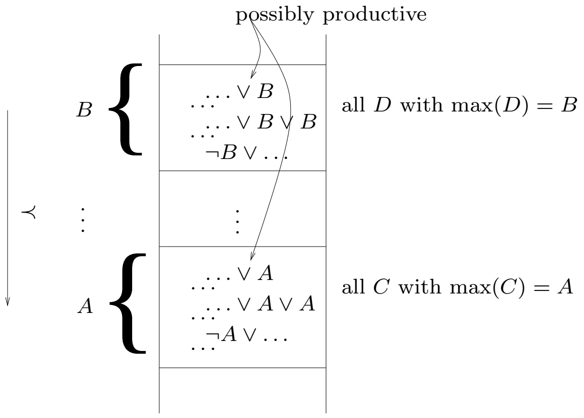
$$\Delta_C := \begin{cases} \{A\}, & \text{if } C \in N,\ C = C' \vee A,\ A \succ C',\ I_C \not\models C \\ \emptyset, & \text{otherwise} \end{cases}$$

We say that C *produces* A, if $\Delta_C = \{A\}$.

The *candidate interpretation* for N (w. r. t. $\succ$) is given as $I_N^{\succ} := \bigcup_C \Delta_C$. (We also simply write I_N or I for $I_N^{\succ}$ if $\succ$ is either irrelevant or known from the context.)

Structure of $N, \succ$

Let $A \succ B$; producing a new atom does not affect smaller clauses.



Some Properties of the Construction

Proposition 2.23

(i) $C = \neg A \vee C' \;\Rightarrow\;$ *no* $D \succeq C$ *produces* A.

(ii) C *productive* $\Rightarrow I_C \cup \Delta_C \models C$.

(iii) *Let* $D' \succ D \succeq C$. *Then*

$$I_D \cup \Delta_D \models C \Rightarrow I_{D'} \cup \Delta_{D'} \models C \text{ and } I_N \models C.$$

If, in addition, $C \in N$ *or* $\max(D) \succ \max(C)$*:*

$$I_D \cup \Delta_D \not\models C \Rightarrow I_{D'} \cup \Delta_{D'} \not\models C \text{ and } I_N \not\models C.$$

(iv) *Let* $D' \succ D \succ C$. *Then*

$$I_D \models C \Rightarrow I_{D'} \models C \text{ and } I_N \models C.$$

If, in addition, $C \in N$ *or* $\max(D) \succ \max(C)$*:*

$$I_D \not\models C \Rightarrow I_{D'} \not\models C \text{ and } I_N \not\models C.$$

(v) $D = C \vee A$ *produces* $A \Rightarrow I_N \not\models C$.

Model Existence Theorem

Theorem 2.24 (Bachmair & Ganzinger 1990) *Let $\succ$ be a clause ordering, let N be saturated w. r. t. Res, and suppose that $\bot \notin N$. Then $I_N^{\succ} \models N$.*

Corollary 2.25 *Let N be saturated w. r. t. Res. Then $N \models \bot \Leftrightarrow \bot \in N$.*

Proof of Theorem 2.24. Suppose $\bot \notin N$, but $I_N^{\succ} \not\models N$. Let $C \in N$ minimal (in $\succ$) such that $I_N^{\succ} \not\models C$. Since C is false in I_N, C is not productive. As $C \neq \bot$ there exists a maximal atom A in C.

Case 1: $C = \neg A \vee C'$ (i. e., the maximal atom occurs negatively)
$\Rightarrow I_N \models A$ and $I_N \not\models C'$
$\Rightarrow$ some $D = D' \vee A \in N$ produces A. As $\frac{D' \vee A \qquad \neg A \vee C'}{D' \vee C'}$, we infer that $D' \vee C' \in N$, and $C \succ D' \vee C'$ and $I_N \not\models D' \vee C'$
$\Rightarrow$ contradicts minimality of C.

Case 2: $C = C' \vee A \vee A$. Then $\frac{C' \vee A \vee A}{C' \vee A}$ yields a smaller counterexample $C' \vee A \in N$. $\Rightarrow$ contradicts minimality of C. □

Compactness of Propositional Logic

Theorem 2.26 (Compactness) *Let N be a set of propositional formulas. Then N is unsatisfiable, if and only if some finite subset $M \subseteq N$ is unsatisfiable.*

Proof. "$\Leftarrow$": trivial.

"$\Rightarrow$": Let N be unsatisfiable.
$\Rightarrow Res^*(N)$ unsatisfiable
$\Rightarrow \bot \in Res^*(N)$ by refutational completeness of resolution
$\Rightarrow \exists n \geq 0 : \bot \in Res^n(N)$
$\Rightarrow \bot$ has a finite resolution proof P;
choose M as the set of assumptions in P. □