2.12 Ordered Resolution with Selection

Motivation: Search space for *Res* *very* large.

Ideas for improvement:

1. In the completeness proof (Model Existence Theorem 2.23) one only needs to resolve and factor maximal atoms
   $\Rightarrow$ if the calculus is restricted to inferences involving maximal atoms, the proof remains correct
   $\Rightarrow$ *order restrictions*

2. In the proof, it does not really matter with which negative literal an inference is performed
   $\Rightarrow$ choose a negative literal don't-care-nondeterministically
   $\Rightarrow$ *selection*

Selection Functions

A selection function is a mapping

$$S : C \quad \mapsto \quad \text{set of occurrences of } \textit{negative} \text{ literals in } C$$

Example of selection with selected literals indicated as $\boxed{X}$:

$$\boxed{\neg A} \vee \neg A \vee B$$

$$\boxed{\neg B_0} \vee \boxed{\neg B_1} \vee A$$

Resolution Calculus $Res_S^{\succ}$

In the completeness proof, we talk about (strictly) maximal literals of *ground* clauses.

In the non-ground calculus, we have to consider those literals that correspond to (strictly) maximal literals of ground instances:

Let $\succ$ be a total and well-founded ordering on ground atoms. A literal L is called [strictly] maximal in a clause C if and only if there exists a ground substitution σ such that for all L' in C: $L\sigma \succeq L'\sigma$ [$L\sigma \succ L'\sigma$].

Resolution Calculus $Res_S^{\succ}$

Let $\succ$ be an atom ordering and S a selection function.

$$\frac{C \lor A \qquad \neg B \lor D}{(C \lor D)\sigma}$$ [ordered resolution with selection]

if $\sigma = \mathrm{mgu}(A, B)$ and

(i) $A\sigma$ strictly maximal wrt. $C\sigma$;

(ii) nothing is selected in C by S;

(iii) either $\neg B$ is selected,
or else nothing is selected in $\neg B \lor D$ and $\neg B\sigma$ is maximal in $D\sigma$.

Resolution Calculus $Res_S^{\succ}$

$$\frac{C \vee A \vee B}{(C \vee A)\sigma} \qquad \text{[ordered factoring]}$$

if $\sigma = \text{mgu}(A, B)$ and $A\sigma$ is maximal in $C\sigma$ and nothing is selected in C.

Special Case: Propositional Logic

For ground clauses the resolution inference simplifies to

$$\frac{C \vee A \qquad D \vee \neg A}{C \vee D}$$

if

(i) $A \succ C$;

(ii) nothing is selected in C by. S;

(iii) $\neg A$ is selected in $D \vee \neg A$,
or else nothing is selected in $D \vee \neg A$ and $\neg A \succeq \max(D)$.

Note: For positive literals, $A \succ C$ is the same as $A \succ \max(C)$.

Search Spaces Become Smaller

1	$A \lor B$	
2	$A \lor \boxed{\neg B}$	
3	$\neg A \lor B$	
4	$\neg A \lor \boxed{\neg B}$	
5	$B \lor B$	Res 1, 3
6	B	Fact 5
7	$\neg A$	Res 6, 4
8	A	Res 6, 2
9	$\bot$	Res 8, 7

we assume $A \succ B$ and S as indicated by $\boxed{X}$. The maximal literal in a clause is depicted in red.

With this ordering and selection function the refutation proceeds strictly deterministically in this example. Generally, proof search will still be non-deterministic but the search space will be much smaller than with unrestricted resolution.

Avoiding Rotation Redundancy

From

$$\dfrac{\dfrac{C_1 \vee A \quad C_2 \vee \neg A \vee B}{C_1 \vee C_2 \vee B} \qquad C_3 \vee \neg B}{C_1 \vee C_2 \vee C_3}$$

we can obtain by rotation

$$\dfrac{C_1 \vee A \qquad \dfrac{C_2 \vee \neg A \vee B \quad C_3 \vee \neg B}{C_2 \vee \neg A \vee C_3}}{C_1 \vee C_2 \vee C_3}$$

another proof of the same clause. In large proofs many rotations are possible. However, if $A \succ B$, then the second proof does not fulfill the orderings restrictions.

Avoiding Rotation Redundancy

Conclusion: In the presence of orderings restrictions (however one chooses $\succ$) no rotations are possible. In other words, orderings identify exactly one representant in any class of of rotation-equivalent proofs.

Lifting Lemma for $Res_S^{\succ}$

Lemma 2.39:

Let C and D be variable-disjoint clauses. If

$$\begin{array}{ccc} C & & D \\ \downarrow \sigma & & \downarrow \rho \\ \dfrac{C\sigma \qquad\qquad D\rho}{C'} & & \end{array} \qquad \text{[propositional inference in } Res_S^{\succ}\text{]}$$

and if $S(C\sigma) \simeq S(C)$, $S(D\rho) \simeq S(D)$ (that is, "corresponding" literals are selected), then there exists a substitution τ such that

$$\dfrac{C \qquad D}{C''} \qquad \text{[Inference in } Res_S^{\succ}\text{]}$$

$$\downarrow \tau$$

$$C' = C''\tau$$

Lifting Lemma for $Res_S^{\succ}$

An analogous lifting lemma holds for factorization.

Saturation of General Clause Sets

Corollary 2.40:

Let N be a set of general clauses saturated under $Res_S^{\succ}$, i.e. $Res_S^{\succ}(N) \subseteq N$. Then there exists a selection function S' such that $S|_N = S'|_N$ and $G_\Sigma(N)$ is also saturated, i.e.,

$$Res_{S'}^{\succ}(G_\Sigma(N)) \subseteq G_\Sigma(N).$$

Proof:

We first define the selection function S' such that $S'(C) = S(C)$ for all clauses $C \in G_\Sigma(N) \cap N$. For $C \in G_\Sigma(N) \setminus N$ we choose a fixed but arbitrary clause $D \in N$ with $C \in G_\Sigma(D)$ and define $S'(C)$ to be those occurrences of literals that are ground instances of the occurrences selected by S in D. Then proceed as in the proof of Corollary 2.32 using the above lifting lemma.

Soundness and Refutational Completeness

Theorem 2.41:
Let $\succ$ be an atom ordering and S a selection function such that $Res_S^{\succ}(N) \subseteq N$. Then

$$N \models \bot \Leftrightarrow \bot \in N$$

Proof:
The "$\Leftarrow$" part is trivial. For the "$\Rightarrow$" part consider first the propositional level: Construct a candidate model I_N as for unrestricted resolution, except that clauses C in N that have selected literals are not productive, even when they are false in I_C and when their maximal atom occurs only once and positively. The result for general clauses follows using Corollary 2.40.

Craig-Interpolation

A theoretical application of ordered resolution is Craig-Interpolation:

Theorem 2.42 (Craig 57):
Let F and G be two propositional formulas such that $F \models G$. Then there exists a formula H (called the interpolant for $F \models G$), such that H contains only prop. variables occurring both in F and in G, and such that $F \models H$ and $H \models G$.

Craig-Interpolation

Proof:

Translate F and $\neg G$ into CNF. let N and M, resp., denote the resulting clause set. Choose an atom ordering $\succ$ for which the prop. variables that occur in F but not in G are maximal. Saturate N into N^* wrt. $Res_S^\succ$ with an empty selection function S . Then saturate $N^* \cup M$ wrt. $Res_S^\succ$ to derive $\bot$. As N^* is already saturated, due to the ordering restrictions only inferences need to be considered where premises, if they are from N^*, only contain symbols that also occur in G. The conjunction of these premises is an interpolant H. The theorem also holds for first-order formulas. For universal formulas the above proof can be easily extended. In the general case, a proof based on resolution technology is more complicated because of Skolemization.

Redundancy

So far: local restrictions of the resolution inference rules using orderings and selection functions.

Is it also possible to delete clauses altogether?
Under which circumstances are clauses unnecessary?
(Conjecture: e. g., if they are tautologies or if they are subsumed by other clauses.)

Intuition: If a clause is guaranteed to be neither a minimal counterexample nor productive, then we do not need it.

A Formal Notion of Redundancy

Let N be a set of ground clauses and C a ground clause (not necessarily in N). C is called redundant w. r. t. N, if there exist $C_1, \dots, C_n \in N$, $n \geq 0$, such that $C_i \prec C$ and $C_1, \dots, C_n \models C$.

Redundancy for general clauses:
C is called redundant w. r. t. N, if all ground instances $C\sigma$ of C are redundant w. r. t. $G_\Sigma(N)$.

Intuition: Redundant clauses are neither minimal counterexamples nor productive.

Note: The same ordering $\succ$ is used for ordering restrictions and for redundancy (and for the completeness proof).

Examples of Redundancy

Proposition 2.43:

- C tautology (i.e., $\models C$) $\Rightarrow$ C redundant w. r. t. any set N.
- $C\sigma \subset D \;\Rightarrow\; D$ redundant w. r. t. $N \cup \{C\}$
- $C\sigma \subseteq D \;\Rightarrow\; D \vee \overline{L}\sigma$ redundant w. r. t. $N \cup \{C \vee L,\ D\}$

(Under certain conditions one may also use non-strict subsumption, but this requires a slightly more complicated definition of redundancy.)

Saturation up to Redundancy

N is called saturated up to redundancy (wrt. $Res_S^{\succ}$)

$$:\Leftrightarrow Res_S^{\succ}(N \setminus Red(N)) \subseteq N \cup Red(N)$$

Theorem 2.44:

Let N be saturated up to redundancy. Then

$$N \models \bot \Leftrightarrow \bot \in N$$

Saturation up to Redundancy

Proof (Sketch):

(i) Ground case:

- consider the construction of the candidate model $I_N^{\succ}$ for $Res_S^{\succ}$
- redundant clauses are not productive
- redundant clauses in N are not minimal counterexamples for $I_N^{\succ}$

The premises of "essential" inferences are either minimal counterexamples or productive.

(ii) Lifting: no additional problems over the proof of Theorem 2.41.

Monotonicity Properties of Redundancy

Theorem 2.45:

(i) $N \subseteq M \Rightarrow Red(N) \subseteq Red(M)$

(ii) $M \subseteq Red(N) \Rightarrow Red(N) \subseteq Red(N \setminus M)$

Proof: Exercise.

We conclude that redundancy is preserved when, during a theorem proving process, one adds (derives) new clauses or deletes redundant clauses.

A Resolution Prover

So far: static view on completeness of resolution:

Saturated sets are inconsistent if and only if they contain $\bot$.

We will now consider a dynamic view:

How can we get saturated sets in practice?

The theorems 2.44 and 2.45 are the basis for the completeness proof of our prover RP.

Rules for Simplifications and Deletion

We want to employ the following rules for simplification of prover states N:

- Deletion of tautologies

$$N \cup \{C \vee A \vee \neg A\} \;\triangleright\; N$$

- Deletion of subsumed clauses

$$N \cup \{C, D\} \;\triangleright\; N \cup \{C\}$$

  if $C\sigma \subseteq D$ (C subsumes D).

- Reduction (also called subsumption resolution)

$$N \cup \{C \vee L,\;\; D \vee C\sigma \vee \overline{L}\sigma\} \;\triangleright\; N \cup \{C \vee L, D \vee C\sigma\}$$

Resolution Prover *RP*

3 clause sets: N(ew) containing new resolvents

P(rocessed) containing simplified resolvents

clauses get into O(ld) once their inferences have been computed

Strategy: Inferences will only be computed when there are no possibilities for simplification

Transition Rules for *RP* (I)

Tautology elimination

$$\boldsymbol{N} \cup \{C\} \mid \boldsymbol{P} \mid \boldsymbol{O} \quad \triangleright \quad \boldsymbol{N} \mid \boldsymbol{P} \mid \boldsymbol{O}$$

if C is a tautology

Forward subsumption

$$\boldsymbol{N} \cup \{C\} \mid \boldsymbol{P} \mid \boldsymbol{O} \quad \triangleright \quad \boldsymbol{N} \mid \boldsymbol{P} \mid \boldsymbol{O}$$

if some $D \in \boldsymbol{P} \cup \boldsymbol{O}$ subsumes C

Backward subsumption

$$\boldsymbol{N} \cup \{C\} \mid \boldsymbol{P} \cup \{D\} \mid \boldsymbol{O} \quad \triangleright \quad \boldsymbol{N} \cup \{C\} \mid \boldsymbol{P} \mid \boldsymbol{O}$$

$$\boldsymbol{N} \cup \{C\} \mid \boldsymbol{P} \mid \boldsymbol{O} \cup \{D\} \quad \triangleright \quad \boldsymbol{N} \cup \{C\} \mid \boldsymbol{P} \mid \boldsymbol{O}$$

if C strictly subsumes D

Transition Rules for *RP* (II)

Forward reduction

$$\boldsymbol{N} \cup \{C \vee L\} \mid \boldsymbol{P} \mid \boldsymbol{O} \quad \rhd \quad \boldsymbol{N} \cup \{C\} \mid \boldsymbol{P} \mid \boldsymbol{O}$$

if there exists $D \vee L' \in \boldsymbol{P} \cup \boldsymbol{O}$

such that $\overline{L} = L'\sigma$ and $D\sigma \subseteq C$

Backward reduction

$$\boldsymbol{N} \mid \boldsymbol{P} \cup \{C \vee L\} \mid \boldsymbol{O} \quad \rhd \quad \boldsymbol{N} \mid \boldsymbol{P} \cup \{C\} \mid \boldsymbol{O}$$

$$\boldsymbol{N} \mid \boldsymbol{P} \mid \boldsymbol{O} \cup \{C \vee L\} \quad \rhd \quad \boldsymbol{N} \mid \boldsymbol{P} \cup \{C\} \mid \boldsymbol{O}$$

if there exists $D \vee L' \in \boldsymbol{N}$

such that $\overline{L} = L'\sigma$ and $D\sigma \subseteq C$

Transition Rules for *RP* (III)

Clause processing

$\boldsymbol{N} \cup \{C\} \mid \boldsymbol{P} \mid \boldsymbol{O} \quad \rhd \quad \boldsymbol{N} \mid \boldsymbol{P} \cup \{C\} \mid \boldsymbol{O}$

Inference computation

$\emptyset \mid \boldsymbol{P} \cup \{C\} \mid \boldsymbol{O} \quad \rhd \quad \boldsymbol{N} \mid \boldsymbol{P} \mid \boldsymbol{O} \cup \{C\},$

with $\boldsymbol{N} = Res_S^{\succ}(\boldsymbol{O} \cup \{C\})$

Soundness and Completeness

Theorem 2.46:

$$N \models \bot \quad \Leftrightarrow \quad N \mid \emptyset \mid \emptyset \quad \overset{*}{\rhd} \quad N' \cup \{\bot\} \mid \_ \mid \_$$

Proof in

L. Bachmair, H. Ganzinger: Resolution Theorem Proving (on H. Ganzinger's Web page under Publications/Journals; appeared in the Handbook on Automated Theorem Proving, 2001)