2.4 Algorithmic Problems

Validity(F): $\models F$?

Satisfiability(F): F satisfiable?

Entailment(F,G): does F entail G?

Model(A,F): $A \models F$?

Solve(A,F): find an assignment β such that $A, \beta \models F$

Solve(F): find a substitution σ such that $\models F\sigma$

Abduce(F): find G with "certain properties" such that G entails F

Gödel's Famous Theorems

1. For most signatures Σ, validity is undecidable for Σ-formulas. (One can easily encode Turing machines in most signatures.)
2. For each signature Σ, the set of valid Σ-formulas is recursively enumerable.
   (We will prove this by giving complete deduction systems.)
3. For $\Sigma = \Sigma_{PA}$ and $\mathbb{N}_* = (\mathbb{N}, 0, s, +, *)$, the theory $Th(\mathbb{N}_*)$ is not recursively enumerable.

These complexity results motivate the study of subclasses of formulas (fragments) of first-order logic

Q: Can you think of any fragments of first-order logic for which validity is decidable?

Some Decidable Fragments

Some decidable fragments:

- Monadic class: no function symbols, all predicates unary; validity is NEXPTIME-complete.
- Variable-free formulas without equality: satisfiability is NP-complete. (why?)
- Variable-free Horn clauses (clauses with at most one positive atom): entailment is decidable in linear time.
- Finite model checking is decidable in time polynomial in the size of the structure and the formula.

2.5 Normal Forms and Skolemization

Study of normal forms motivated by

- reduction of logical concepts,
- efficient data structures for theorem proving.

The main problem in first-order logic is the treatment of quantifiers. The subsequent normal form transformations are intended to eliminate many of them.

Prenex Normal Form

Prenex formulas have the form

$$Q_1 x_1 \ldots Q_n x_n \; F,$$

where F is quantifier-free and $Q_i \in \{\forall, \exists\}$;
we call $Q_1 x_1 \ldots Q_n x_n$ the quantifier prefix and F the matrix of the formula.

Prenex Normal Form

Computing prenex normal form by the rewrite relation $\Rightarrow_P$:

$$
\begin{array}{rcll}
(F \leftrightarrow G) & \Rightarrow_P & (F \rightarrow G) \wedge (G \rightarrow F) & \\
\neg QxF & \Rightarrow_P & \overline{Q}x\neg F & (\neg Q) \\
(QxF \ \rho \ G) & \Rightarrow_P & Qy(F[y/x] \ \rho \ G), \ y \text{ fresh}, \ \rho \in \{\wedge, \vee\} & \\
(QxF \rightarrow G) & \Rightarrow_P & \overline{Q}y(F[y/x] \rightarrow G), \ y \text{ fresh} & \\
(F \ \rho \ QxG) & \Rightarrow_P & Qy(F \ \rho \ G[y/x]), \ y \text{ fresh}, \ \rho \in \{\wedge, \vee, \rightarrow\} &
\end{array}
$$

Here $\overline{Q}$ denotes the quantifier dual to Q, i.e., $\overline{\forall} = \exists$ and $\overline{\exists} = \forall$.

Skolemization

Intuition: replacement of $\exists y$ by a concrete choice function computing y from all the arguments y depends on.

Transformation $\Rightarrow_S$ (to be applied outermost, *not* in subformulas):

$$\forall x_1, \dots, x_n \exists y F \quad \Rightarrow_S \quad \forall x_1, \dots, x_n F[f(x_1, \dots, x_n)/y]$$

where f/n is a new function symbol (Skolem function).

Skolemization

Together: $F \overset{*}{\Rightarrow}_P \underbrace{G}_{\text{prenex}} \overset{*}{\Rightarrow}_S \underbrace{H}_{\text{prenex, no } \exists}$

Theorem 2.9

Let F, G, and H as defined above and closed. Then

(i) F and G are equivalent.

(ii) $H \models G$ but the converse is not true in general.

(iii) G satisfiable (wrt. Σ-Alg) $\Leftrightarrow$ H satisfiable (wrt. Σ'-Alg) where $\Sigma' = (\Omega \cup SKF, \Pi)$, if $\Sigma = (\Omega, \Pi)$.

Clausal Normal Form (Conjunctive Normal Form)

$$
\begin{aligned}
(F \leftrightarrow G) &\quad \Rightarrow_K \quad (F \rightarrow G) \land (G \rightarrow F) \\
(F \rightarrow G) &\quad \Rightarrow_K \quad (\neg F \lor G) \\
\neg(F \lor G) &\quad \Rightarrow_K \quad (\neg F \land \neg G) \\
\neg(F \land G) &\quad \Rightarrow_K \quad (\neg F \lor \neg G) \\
\neg\neg F &\quad \Rightarrow_K \quad F \\
(F \land G) \lor H &\quad \Rightarrow_K \quad (F \lor H) \land (G \lor H) \\
(F \land \top) &\quad \Rightarrow_K \quad F \\
(F \land \bot) &\quad \Rightarrow_K \quad \bot \\
(F \lor \top) &\quad \Rightarrow_K \quad \top \\
(F \lor \bot) &\quad \Rightarrow_K \quad F
\end{aligned}
$$

These rules are to be applied modulo associativity and commutativity of $\land$ and $\lor$. The first five rules, plus the rule $(\neg Q)$, compute the negation normal form (NNF) of a formula.

The Complete Picture

$$
\begin{aligned}
F \quad & \Rightarrow^*_P \quad Q_1 y_1 \ldots Q_n y_n \ G && (G \text{ quantifier-free}) \\
& \Rightarrow^*_S \quad \forall x_1, \ldots, x_m \ H && (m \leq n,\ H \text{ quantifier-free}) \\
& \Rightarrow^*_K \quad \underbrace{\underbrace{\forall x_1, \ldots, x_m}_{\text{leave out}} \ \bigwedge_{i=1}^{k} \underbrace{\bigvee_{j=1}^{n_i} L_{ij}}_{\text{clauses } C_i}}_{F'}
\end{aligned}
$$

$N = \{C_1, \ldots, C_k\}$ is called the clausal (normal) form (CNF) of F.

Note: the variables in the clauses are implicitly universally quantified.

The Complete Picture

Theorem 2.10

Let F be closed. Then $F' \models F$.

(The converse is not true in general.)

Theorem 2.11

Let F be closed. Then F is satisfiable iff F' is satisfiable iff N is satisfiable

Optimization

Here is lots of room for optimization since we only can preserve satisfiability anyway:

- size of the CNF exponential when done naively;
- want to preserve the original formula structure;
- want small arity of Skolem functions.

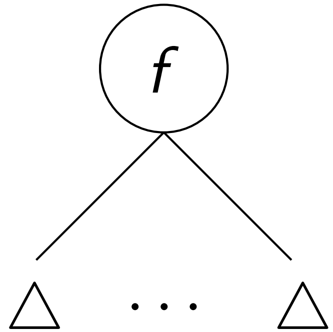
2.6 Herbrand Interpretations

From now an we shall consider PL without equality. Ω shall contains at least one constant symbol.

A Herbrand interpretation (over Σ) is a Σ-algebra $\mathcal{A}$ such that

- $U_{\mathcal{A}} = \mathsf{T}_{\Sigma}$ ($=$ the set of ground terms over Σ)
- $f_{\mathcal{A}} : (s_1, \ldots, s_n) \mapsto f(s_1, \ldots, s_n)$, $f/n \in \Omega$

$$f_{\mathcal{A}}(\triangle, \ldots, \triangle) =$$



Herbrand Interpretations

In other words, *values are fixed* to be ground terms and *functions are fixed* to be the term constructors. Only predicate symbols $p/m \in \Pi$ may be freely interpreted as relations $p_{\mathcal{A}} \subseteq \mathsf{T}_{\Sigma}^{m}$.

Proposition 2.12
Every set of ground atoms I uniquely determines a Herbrand interpretation $\mathcal{A}$ via

$$(s_1, \ldots, s_n) \in p_{\mathcal{A}} \quad :\Leftrightarrow \quad p(s_1, \ldots, s_n) \in I$$

Thus we shall identify Herbrand interpretations (over Σ) with sets of Σ-ground atoms.

Herbrand Interpretations

Example: $\Sigma_{Pres} = (\{0/0, s/1, +/2\},\ \{</2, \leq/2\})$

$\mathbb{N}$ as Herbrand interpretation over Σ_{Pres}:

$$
\begin{aligned}
I = \{ \quad & 0 \leq 0,\ 0 \leq s(0),\ 0 \leq s(s(0)),\ \ldots, \\
& 0 + 0 \leq 0,\ 0 + 0 \leq s(0),\ \ldots, \\
& \ldots,\ (s(0) + 0) + s(0) \leq s(0) + (s(0) + s(0)) \\
& \ldots \\
& s(0) + 0 < s(0) + 0 + 0 + s(0) \\
& \ldots\}
\end{aligned}
$$

Existence of Herbrand Models

A Herbrand interpretation I is called a Herbrand model of F, if $I \models F$.

Theorem 2.13
Let N be a set of Σ-clauses.

$$\begin{aligned} N \text{ satisfiable} \quad &\Leftrightarrow \quad N \text{ has a Herbrand model (over } \Sigma) \\ &\Leftrightarrow \quad G_\Sigma(N) \text{ has a Herbrand model (over } \Sigma) \end{aligned}$$

where $G_\Sigma(N) = \{C\sigma \text{ ground clause} \mid C \in N,\ \sigma : X \to \mathsf{T}_\Sigma\}$ is the set of ground instances of N.

[The proof will be given below in the context of the completeness proof for resolution.]

Example of a G_Σ

For Σ_{Pres} one obtains for

$$C = (x < y) \vee (y \leq s(x))$$

the following ground instances:

$(0 < 0) \vee (0 \leq s(0))$

$(s(0) < 0) \vee (0 \leq s(s(0)))$

$\ldots$

$(s(0) + s(0) < s(0) + 0) \vee (s(0) + 0 \leq s(s(0) + s(0)))$

$\ldots$

2.7 Inference Systems and Proofs

Inference systems Γ (proof calculi) are sets of tuples

$$(F_1, \dots, F_n, F_{n+1}), \;\; n \geq 0,$$

called inferences or inference rules, and written

$$\frac{\overbrace{F_1 \;\; \dots \;\; F_n}^{\text{premises}}}{\underbrace{F_{n+1}}_{\text{conclusion}}} \;.$$

Clausal inference system: premises and conclusions are clauses. One also considers inference systems over other data structures (cf. below).

Proofs

A proof in Γ of a formula F from a a set of formulas N (called assumptions) is a sequence $F_1, \ldots, F_k$ of formulas where

(i) $F_k = F$,

(ii) for all $1 \leq i \leq k$: $F_i \in N$, or else there exists an inference $(F_{i_1}, \ldots, F_{i_{n_i}}, F_i)$ in Γ, such that $0 \leq i_j < i$, for $1 \leq j \leq n_i$.

Soundness and Completeness

Provability $\vdash_\Gamma$ of F from N in Γ:

$N \vdash_\Gamma F \;\; :\Leftrightarrow$ there exists a proof Γ of F from N.

Γ is called sound $:\Leftrightarrow$

$$\frac{F_1 \;\; \ldots \;\; F_n}{F} \in \Gamma \quad \Rightarrow \quad F_1, \ldots, F_n \models F$$

Γ is called complete $:\Leftrightarrow$

$$N \models F \quad \Rightarrow \quad N \vdash_\Gamma F$$

Γ is called refutationally complete $:\Leftrightarrow$

$$N \models \bot \quad \Rightarrow \quad N \vdash_\Gamma \bot$$

Soundness and Completeness

Proposition 2.14

(i) Let Γ be sound. Then $N \vdash_\Gamma F \Rightarrow N \models F$

(ii) $N \vdash_\Gamma F \Rightarrow$ there exist $F_1, \ldots, F_n \in N$ s.t. $F_1, \ldots, F_n \vdash_\Gamma F$ (resembles compactness).

Proofs as Trees

markings	$\widehat{=}$	formulas		
leaves	$\widehat{=}$	assumptions and axioms		
other nodes	$\widehat{=}$	inferences: conclusion	$\widehat{=}$	ancestor
		premises	$\widehat{=}$	direct descendants

$$
\dfrac{P(g(a,b)) \qquad \dfrac{\dfrac{\dfrac{P(f(a)) \vee Q(b) \qquad \dfrac{\dfrac{P(f(a)) \vee Q(b) \quad \neg P(f(a)) \vee \neg P(f(a)) \vee Q(b)}{\neg P(f(a)) \vee Q(b) \vee Q(b)}}{\neg P(f(a)) \vee Q(b)}}{Q(b) \vee Q(b)}}{Q(b)} \qquad \neg P(f(a)) \vee \neg Q(b)}{\neg P(g(a,b))}}{\bot}
$$

2.8 Propositional Resolution

We observe that propositional clauses and ground clauses are the same concept.

In this section we only deal with ground clauses.

The Resolution Calculus *Res*

Resolution inference rule:

$$\frac{C \vee A \qquad \neg A \vee D}{C \vee D}$$

Terminology: $C \vee D$: resolvent; A: resolved atom

(Positive) factorisation inference rule:

$$\frac{C \vee A \vee A}{C \vee A}$$

The Resolution Calculus *Res*

These are schematic inference rules; for each substitution of the schematic variables C, D, and A, respectively, by ground clauses and ground atoms we obtain an inference rule.

As "$\vee$" is considered associative and commutative, we assume that A and $\neg A$ can occur anywhere in their respective clauses.

Sample Refutation

1.	$\neg P(f(a)) \vee \neg P(f(a)) \vee Q(b)$	(given)
2.	$P(f(a)) \vee Q(b)$	(given)
3.	$\neg P(g(b,a)) \vee \neg Q(b)$	(given)
4.	$P(g(b,a))$	(given)
5.	$\neg P(f(a)) \vee Q(b) \vee Q(b)$	(Res. 2. into 1.)
6.	$\neg P(f(a)) \vee Q(b)$	(Fact. 5.)
7.	$Q(b) \vee Q(b)$	(Res. 2. into 6.)
8.	$Q(b)$	(Fact. 7.)
9.	$\neg P(g(b,a))$	(Res. 8. into 3.)
10.	$\bot$	(Res. 4. into 9.)

Resolution with Implicit Factorization *RIF*

$$\frac{C \vee A \vee \ldots \vee A \qquad \neg A \vee D}{C \vee D}$$

1.	$\neg P(f(a)) \vee \neg P(f(a)) \vee Q(b)$	(given)
2.	$P(f(a)) \vee Q(b)$	(given)
3.	$\neg P(g(b,a)) \vee \neg Q(b)$	(given)
4.	$P(g(b,a))$	(given)
5.	$\neg P(f(a)) \vee Q(b) \vee Q(b)$	(Res. 2. into 1.)
6.	$Q(b) \vee Q(b) \vee Q(b)$	(Res. 2. into 5.)
7.	$\neg P(g(b,a))$	(Res. 6. into 3.)
8.	$\bot$	(Res. 4. into 7.)

Soundness of Resolution

Theorem 2.15

Propositional resolution is sound.

Proof:

Let $I \in \Sigma$-Alg. To be shown:

(i) for resolution: $I \models C \vee A,\ I \models D \vee \neg A \;\Rightarrow\; I \models C \vee D$

(ii) for factorization: $I \models C \vee A \vee A \;\Rightarrow\; I \models C \vee A$

ad (i): Assume premises are valid in I. Two cases need to be considered: (a) A is valid, or (b) $\neg A$ is valid.

a) $I \models A \Rightarrow I \models D \Rightarrow I \models C \vee D$

b) $I \models \neg A \Rightarrow I \models C \Rightarrow I \models C \vee D$

ad (ii): even simpler.

Soundness of Resolution

Note: In propositional logic (ground clauses) we have:

1. $I \models L_1 \vee \ldots \vee L_n \;\Leftrightarrow\;$ there exists i: $I \models L_i$.
2. $I \models A$ or $I \models \neg A$.